Graph coloring: Theory and Application

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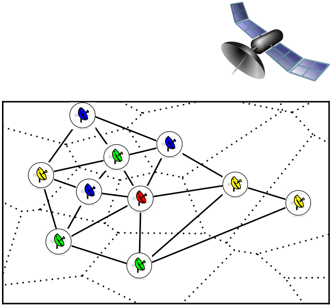
Outline

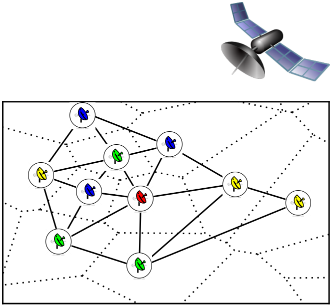
1. Basic definitions
2. Some upper bounds
3. Some lower bounds
4. Coloring planar graphs
5. Coloring edges
6. Concluding remarks

Basic definitions



- Assign frequencies to radio antennas;
- Proximity causes noise;
- Hence, close antennas must get distinct frequencies.

Graph Coloring



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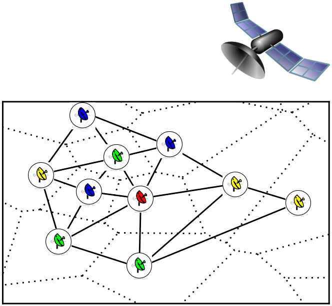

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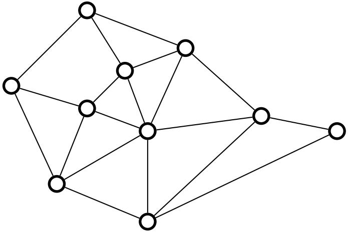
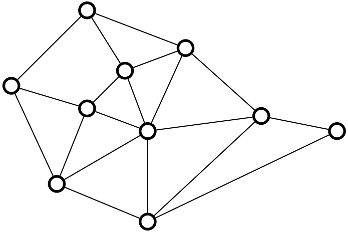


- Pair (V, E) of vertices and edges;
- E is a set of subsets of V of size 2;
- We write $uv \in E(G)$, and say that u, v are adjacent or neighbors.

Graph



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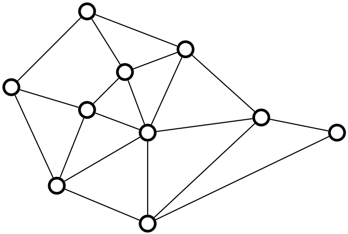

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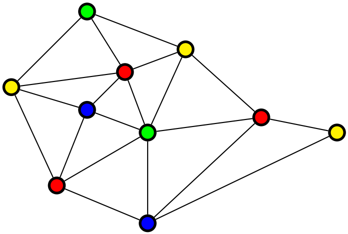
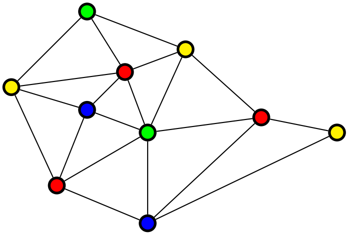


Proper coloring

- $f : V(G) \to [k]$ s.t. $f(u) \neq f(v)$ for every edge $uv \in E(G)$;
- $\chi(G) = \min k$ for which G admits a k-coloring;
- Given G and $k \in \mathbb{N}$, decide $\chi(G) \leq k$.

Proper Coloring



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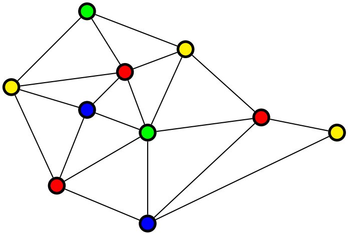

Chromatic number

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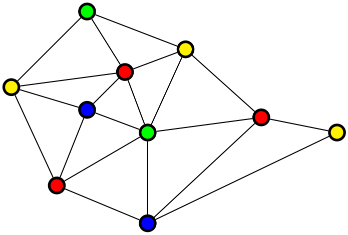

Decision problem

- $f : V(G) \to [k]$ s.t. $f(u) \neq f(v)$ for every edge $uv \in E(G)$;
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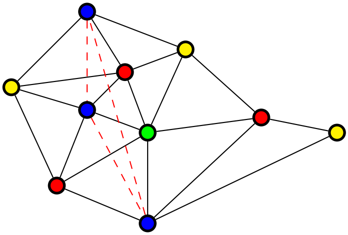
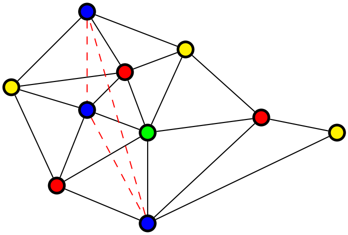
Partition into independent sets



Independent set

- Subset $S \subseteq V(G)$ s.t. $uv \notin E(G)$ for every $u, v \in S$;
- Partition $S_1, \ldots, S_k$ s.t. each S_i is an independent set;
- Independent sets are also called stable sets.

Partition into independent sets



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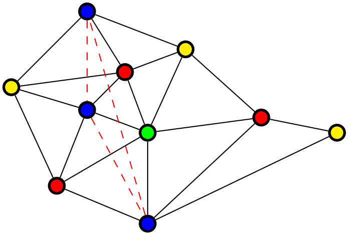

Stable sets

- Subset $S \subseteq V(G)$ s.t. $uv \notin E(G)$ for every $u, v \in S$;
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One of the Karp's 21 Problems

- NP-complete, even if k is fixed, $k \geq 3$;
- Para-NP-complete when parameterized by k;
- Impossible to approximate by a constant factor, unles P $=$ NP.

Richard Karp.
Reducibility among Combinatorial Problems.
In R. E. Miller and J. W. Thatcher (editors). Complexity of Computer Computations. New York: Plenum. pp. 85–103, 1972.

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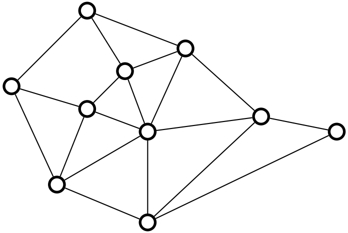
On the hardness of approximating minimization problems.
J. of the ACM 41 (1994) 960–981.

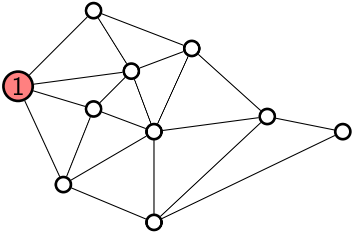
Some upper bounds

Upper bound: maximum degree plus one



Iterate on each $u \in V(G)$, giving smallest color not in $N(u)$.

Upper bound: maximum degree plus one



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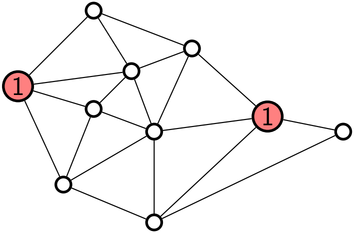

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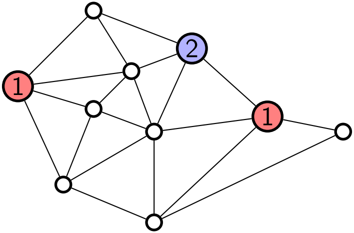

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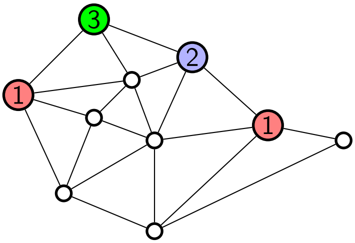

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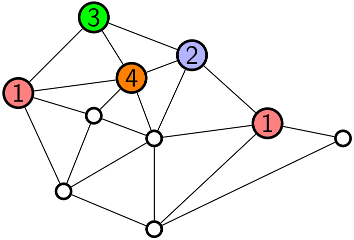

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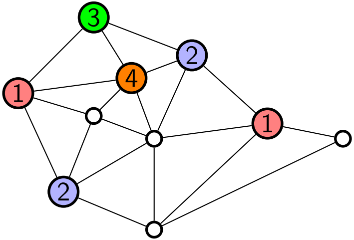

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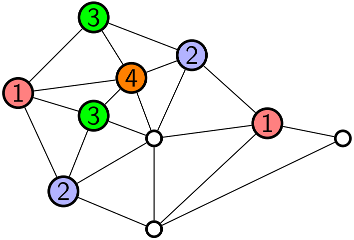

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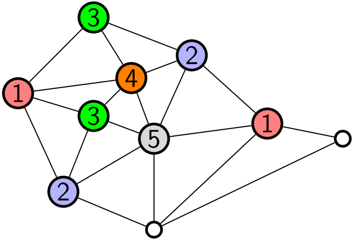

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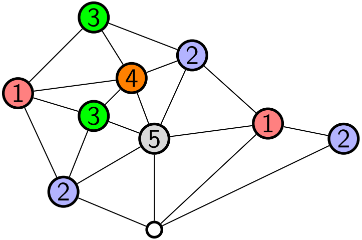

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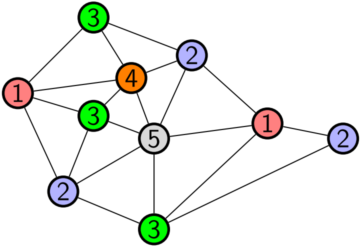

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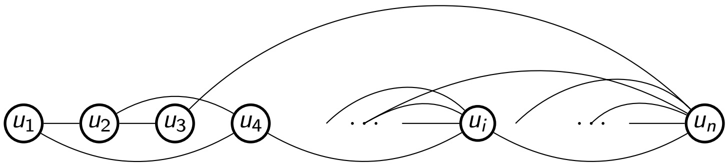

Iterate on the order, giving smallest color not in $N(u_i)$ for each i.

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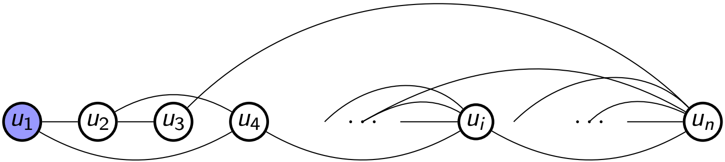

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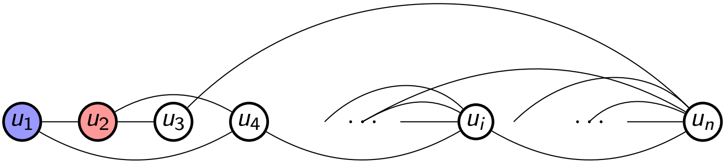

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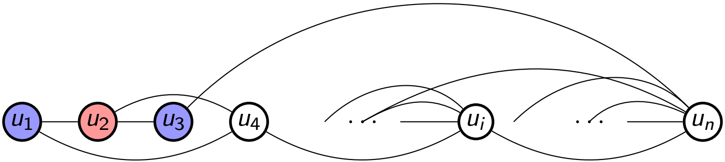

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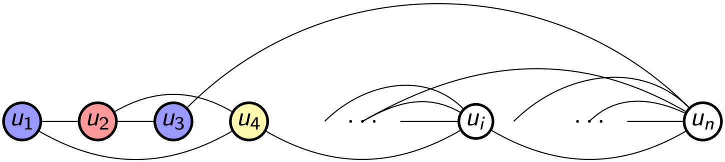

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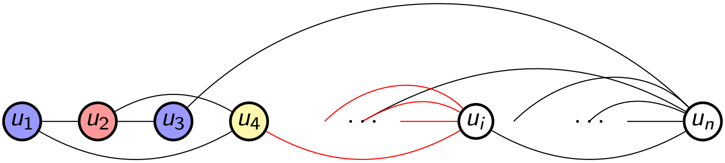

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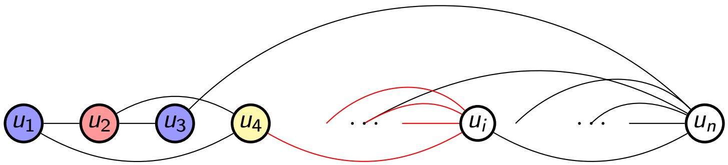
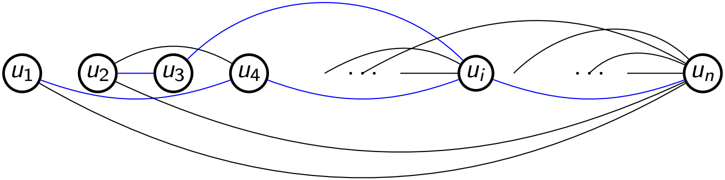


Iterate on the order, giving smallest color not in $N(u_i)$ for each i.

Proposition

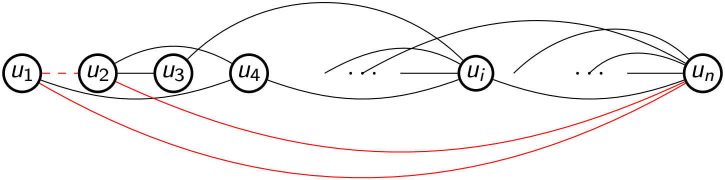
Let $\Delta(G)$ be the maximum degree of G. Then $\chi(G) \leq \Delta(G) + 1$.

Upper bound: maximum degree ~~plus one~~



- For every $i \in \{1, \ldots, n-1\}$, there exists $j > i$ s.t. $u_i u_j$ is an edge; and
- $u_1 u_n$, $u_2 u_n$ are edges, and $u_1 u_2$ is not an edge.

Upper bound: maximum degree ~~plus one~~



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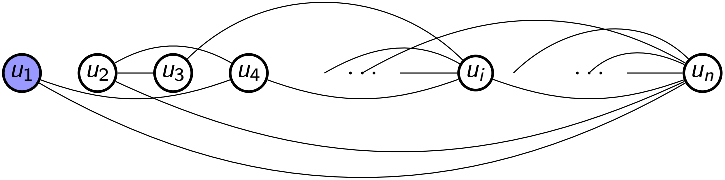

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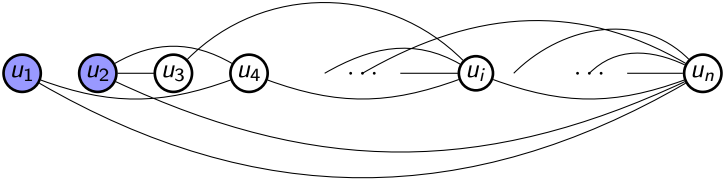

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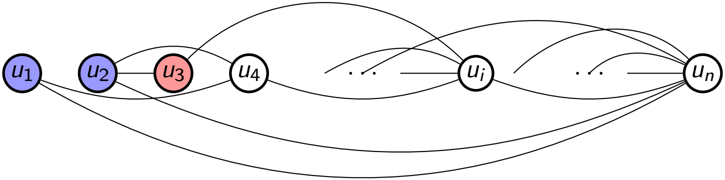

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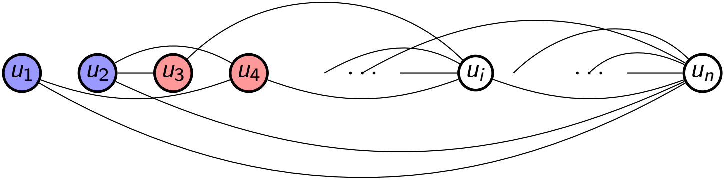

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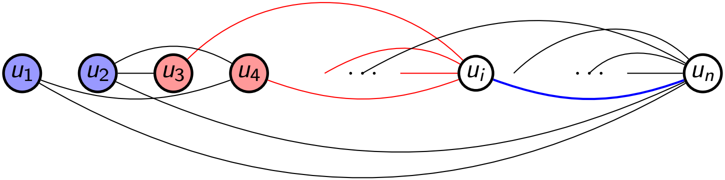

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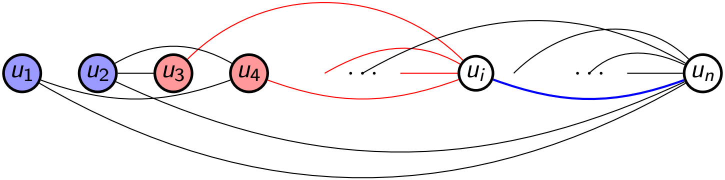

- For every $i \in \{1, \ldots, n-1\}$, there exists $j > i$ s.t. $u_i u_j$ is an edge; and
  $\Rightarrow u_i$ has $\leq d(u_i) - 1$ colored neighbors
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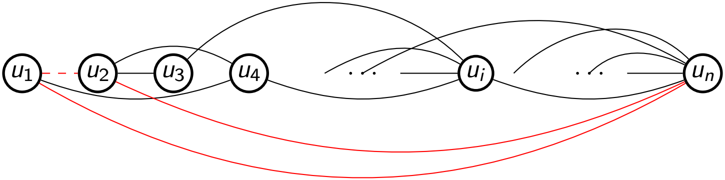

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  $\Rightarrow u_n$ has 2 neighbors of the same color

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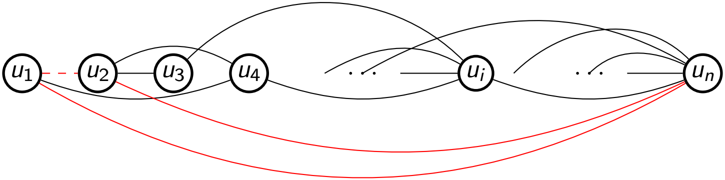

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Obtained coloring uses at most $\Delta(G)$ colors.

Upper bound: maximum degree ~~plus one~~

Proposition

If $(u_1, \ldots, u_n)$ *is an ordering s.t.:*

- *For every* $i \in \{1, \ldots, n-1\}$, *there exists* $j > i$ *s.t.* $u_i u_j$ *is an edge; and*
- $u_1 u_n, u_2 u_n$ *are edges, and* $u_1 u_2$ *is not an edge,*

then $\chi(G) \leq \Delta(G)$.

Theorem (Brooks, 1941)

Such an order exists iff G *is neither an odd cycle, nor a complete graph.*

Upper bound: maximum degree ~~plus one~~

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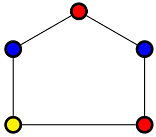

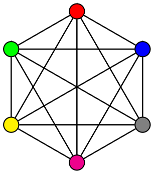

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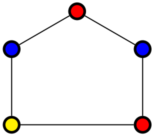
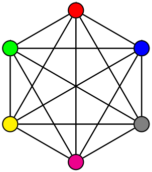
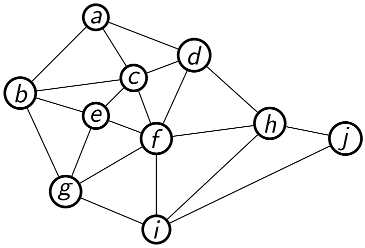
Such an order exists iff G is neither an odd cycle, nor a complete graph.
$\chi(G) = \Delta(G) + 1$ iff G is an odd cycle, or a complete graph.



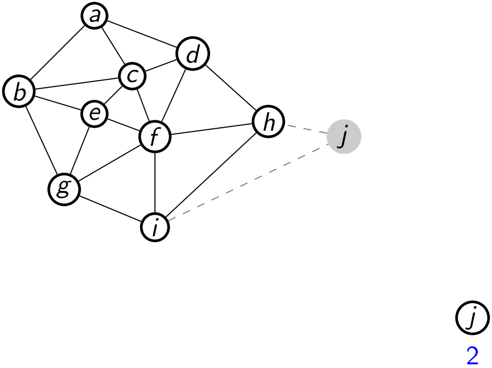


Upper bound: degeneracy



Iteratively put in the beginning the vertex with smallest degree.

Upper bound: degeneracy



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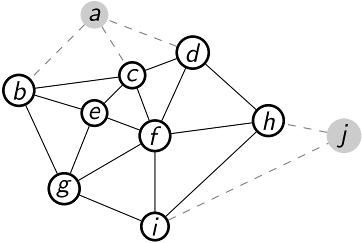

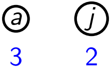

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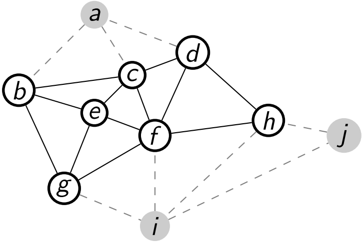

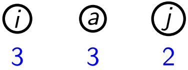

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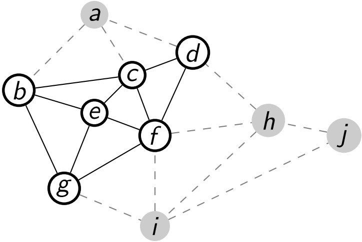

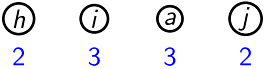

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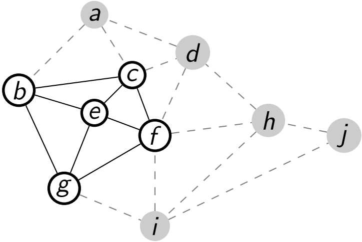

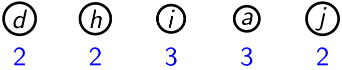

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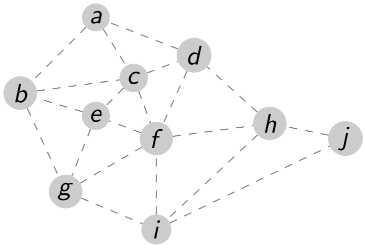

g	e	b	f	c	d	h	i	a	j
0	1	2	2	3	2	2	3	3	2

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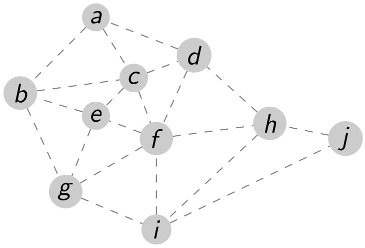

g	e	b	f	c	d	h	i	a	j
0	1	2	2	3	2	2	3	3	2

Iteratively put in the beginning the vertex with smallest degree.

Order that produces a coloring with at most 4 colors.

Upper bound: degeneracy

Degeneracy (or coloring number)

(minimum degree $\delta(H)$ over all subgraphs H of G, plus 1.)

$$col(G) = 1 + \max_{H \subseteq G} \delta(H).$$

Theorem (Szekeres-Wilf, 1968)

$$\chi(G) \leq col(G).$$

Upper bound: degeneracy

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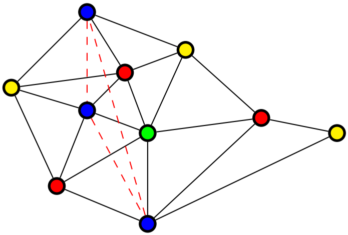
Some lower bounds

Lower bound: n over max independent set

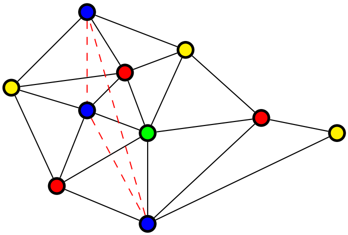


Recall:

Independent set

- Subset $S \subseteq V(G)$ s.t. $uv \notin E(G)$ for every $u, v \in S$;
- Partition $S_1, \ldots, S_k$ s.t. each S_i is an independent set; and

Lower bound: n over max independent set



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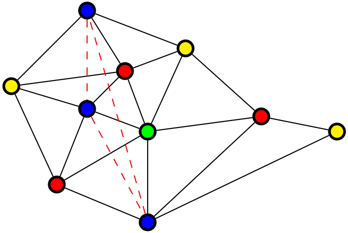

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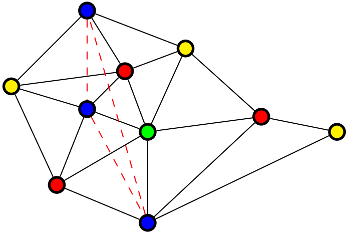
$\theta(G) = \max |S|$ s.t. S is an independent set of G.



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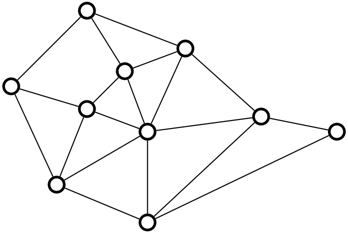
Proposition

For every graph G,

$$\chi(G) \geq \frac{|V(G)|}{\theta(G)}.$$

Lower bound: clique number



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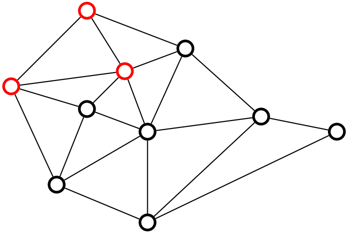

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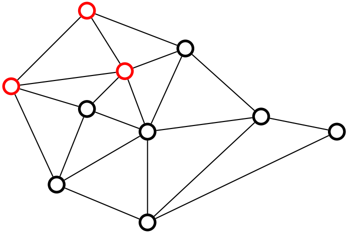
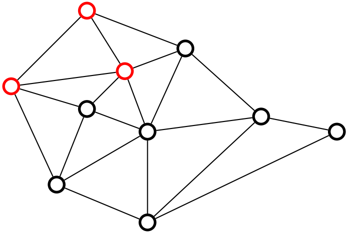


Clique

- Subset $S \subseteq V(G)$ s.t. $uv \in E(G)$ for every $u, v \in S$;
- Maximum size of a clique; denoted by $\omega(G)$.

Lower bound: clique number



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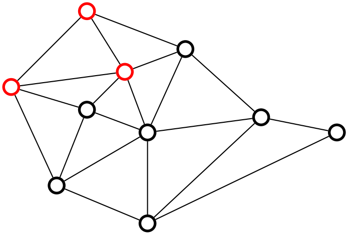

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$$\chi(G) \geq \omega(G).$$

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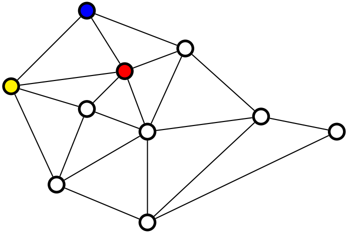

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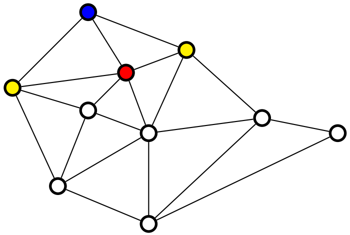

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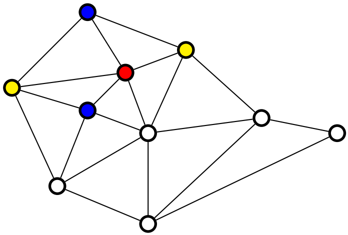

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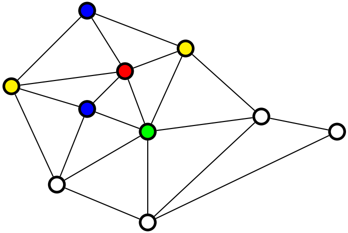

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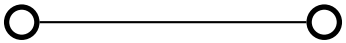
Clique number can be arbitrarily smaller than $\chi(G)$.

Mycielki's construction



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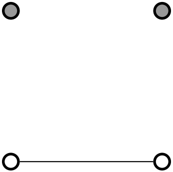

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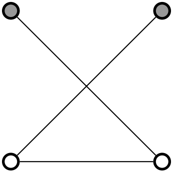

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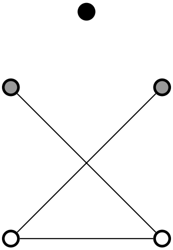

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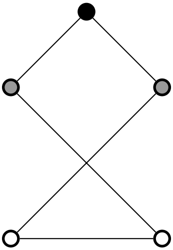

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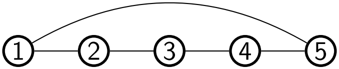

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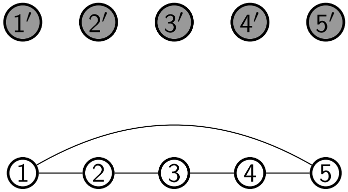

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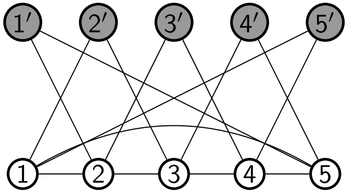

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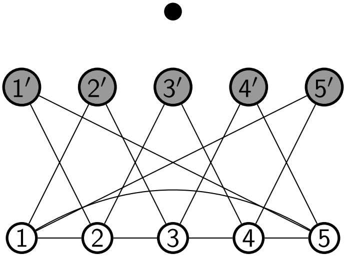

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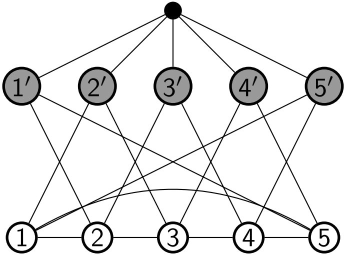

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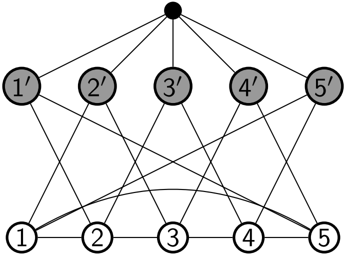
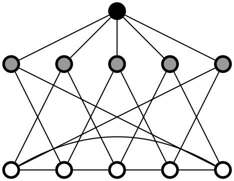


Mycielskian of G

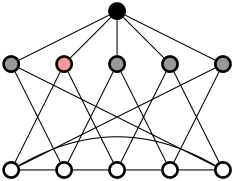
Mycielki's construction



Proposition

If G has no triangles, then the Mycielskian of G also has no triangles.

Mycielki's construction



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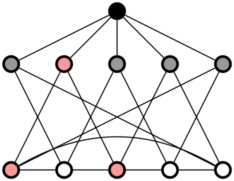

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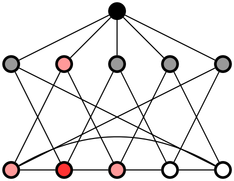

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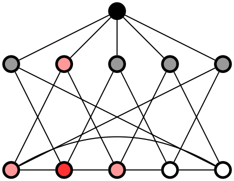

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If $\chi(G) = k$, then $\chi(G') = k + 1$, where G' is the Mycielskian of G.

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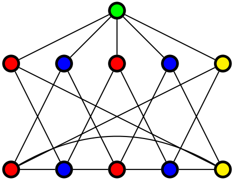

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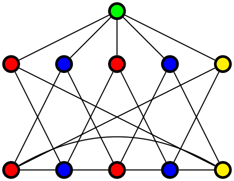

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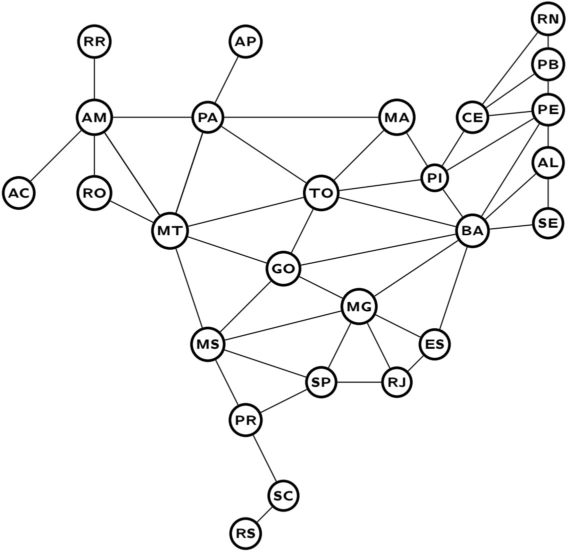
Corollary

For each integer $k \geq 2$, there exists G s.t. $\omega(G) = 2$ and $\chi(G) = k$.

Coloring planar graphs

Coloring a map

Coloring a ~~map~~ planar graph



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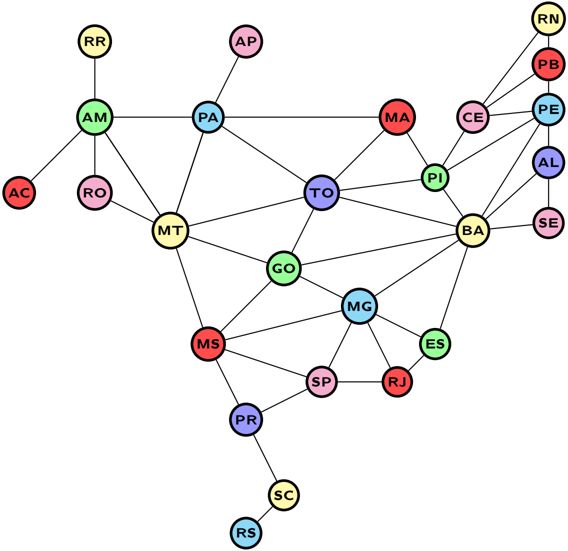

Coloring a ~~map~~ planar graph

Four color Theorem

Conjecture (Guthrie, 1852)

(Wrongfully credited to De Morgan)
Four colors are always enough to color a map.

Theorem (Appel and Haken, 1989)

Sure! (But to do it, we need computers!)

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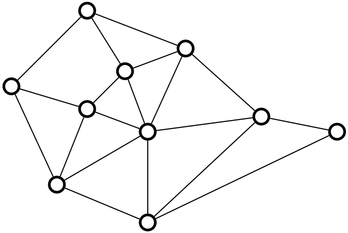

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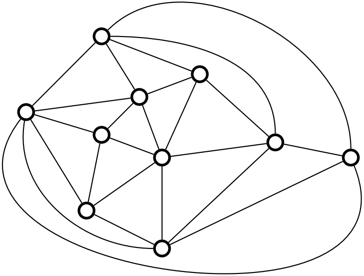

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Suppose G triangulated. We get $\sum_f |f| = 3F = 2m$.
By Euler's Relation ($n - m + F = 2$), we then get:

$$6n - 6m + 6F = 6n - 6m + 4m = 6n - 2m = 12.$$

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It follows that there exists $v \in V(G)$ such that $d(v) < 6$.

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We also know that $2m = \sum_{v \in V(G)} d(v)$. Therefore:

$$6n - \sum_{v \in V(G)} d(v) = \sum_{v \in V(G)} (6 - d(v)) = 12.$$

It follows that there exists $v \in V(G)$ such that $d(v) < 6$. In other words, G is 5-degenerate and hence can be colored with 6 colors.

~~Four~~ Five color theorem

Theorem (Heawood, 1891)

Five colors are always enough to color a map.

By induction on $n = |V(G)|$. If $n \leq 5$, there is nothing to do.
If there exists $u \in V(G)$ with $d(u) < 5$, then

- Apply induction on $G - u$, obtaining f that uses at most 5 colors;
- Choose a color not appearing in $N(u)$ with which to color u.

Otherwise, choose $u \in V(G)$ s.t. $d(u) = 5$ (recall previous slide).

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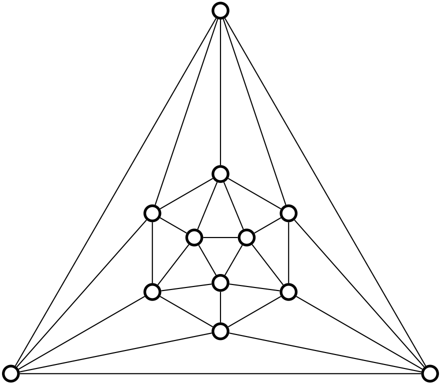

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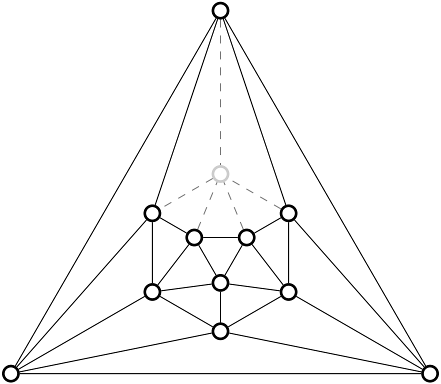

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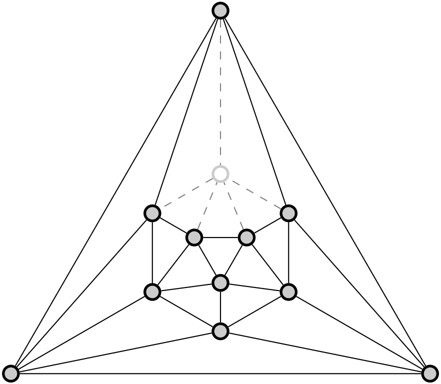

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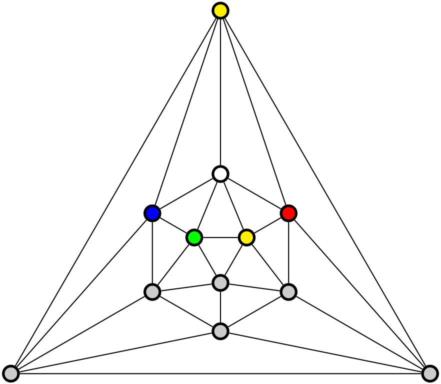

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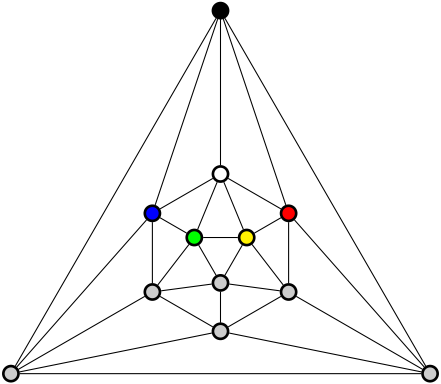

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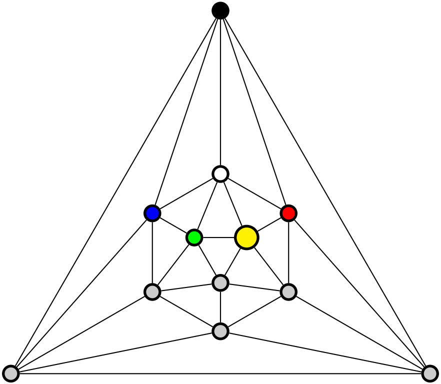

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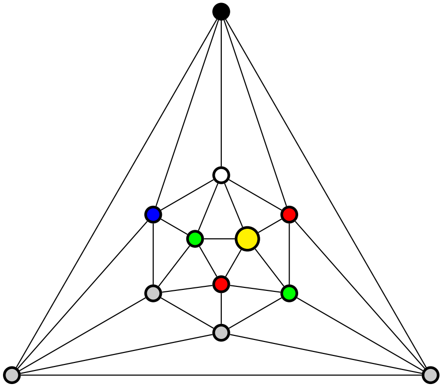

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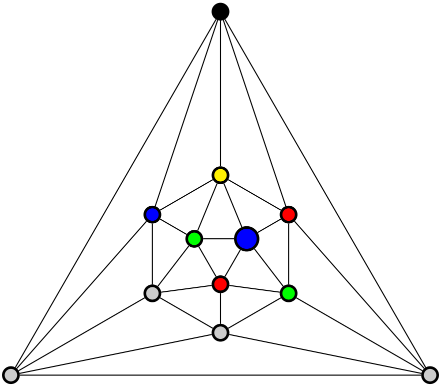

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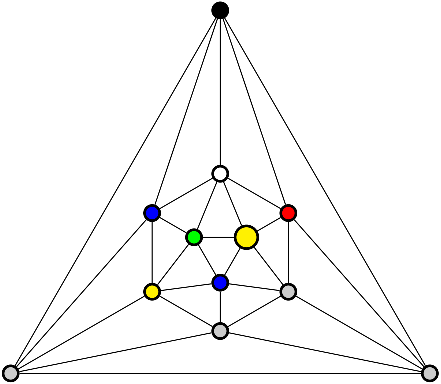

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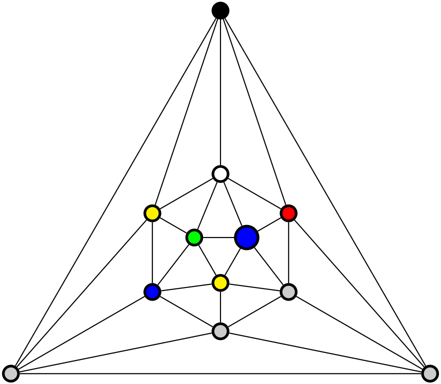

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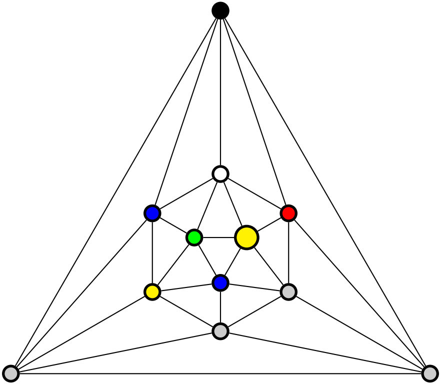

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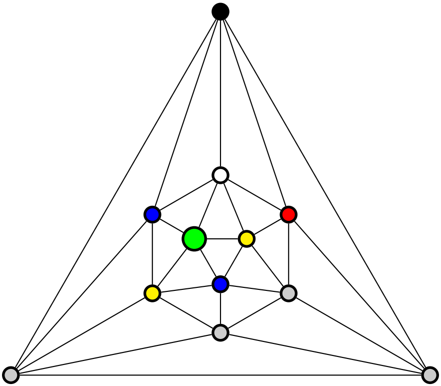

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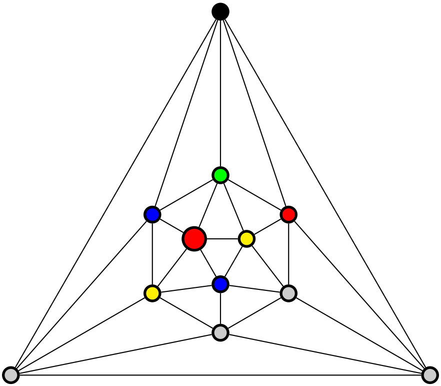

Coloring edges

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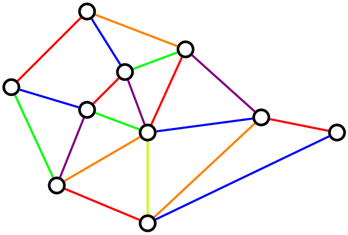
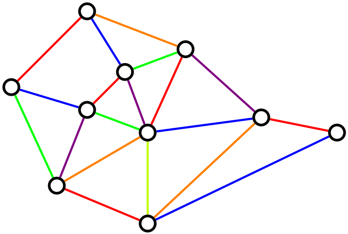


Edge k-coloring

- $f : E(G) \to [k]$ s.t. $f(e) \neq f(e')$ for every adjacent $e, e' \in E(G)$;
- $\chi'(G) = \min k$ for which G admits an edge k-coloring;
- Given G and $k \in \mathbb{N}$, decide $\chi'(G) \leq k$.

Coloring edges



Chromatic index

- $f : E(G) \to [k]$ s.t. $f(e) \neq f(e')$ for every adjacent $e, e' \in E(G)$;
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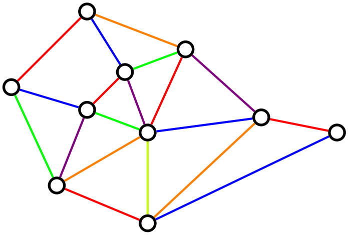
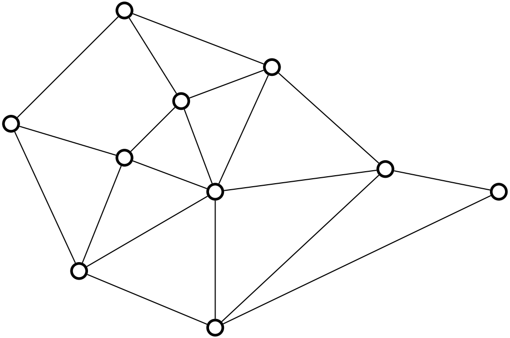


Decision problem

- $f : E(G) \to [k]$ s.t. $f(e) \neq f(e')$ for every adjacent $e, e' \in E(G)$;
- $\chi'(G) = \min k$ for which G admits an edge k-coloring;
- Given G and $k \in \mathbb{N}$, decide $\chi'(G) \leq k$.

Equivalent to coloring line graphs



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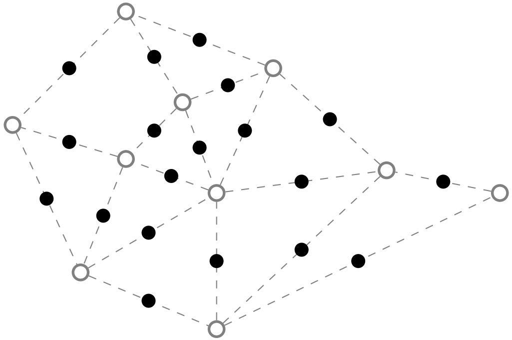

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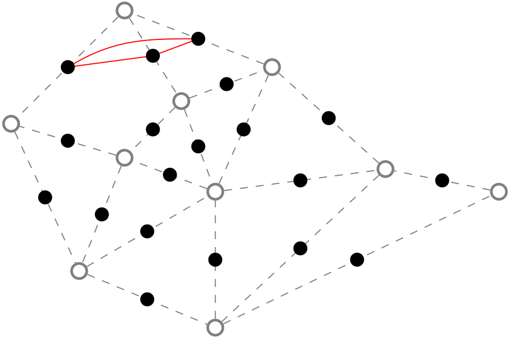

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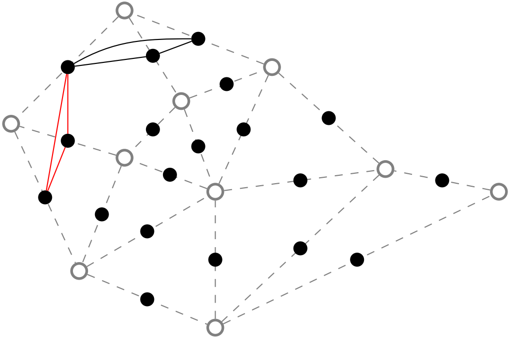

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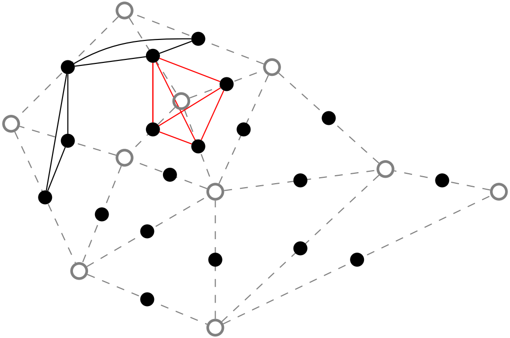

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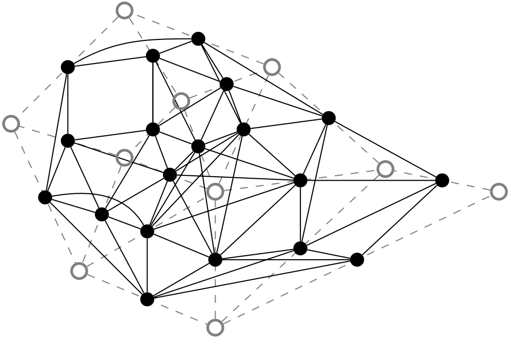

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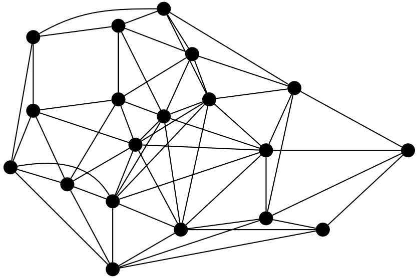
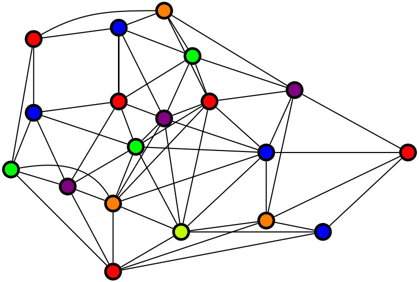


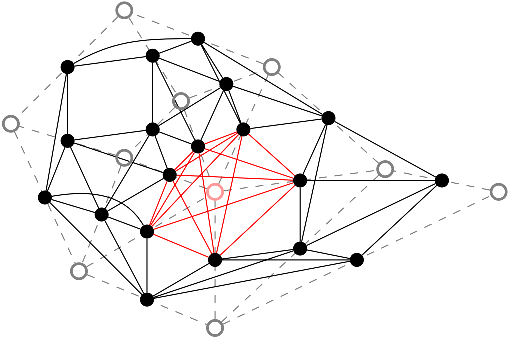
Line graph of G, denoted by $L(G)$.

Equivalent to coloring line graphs



Line graph of G, denoted by $L(G)$.

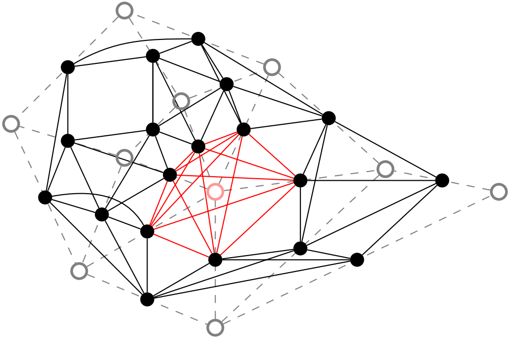
Lower bound: clique number of line graph



Proposition

$$\Delta(G) \leq \omega(L(G)) \leq \chi'(G) \quad .$$

Lower bound: clique number of line graph



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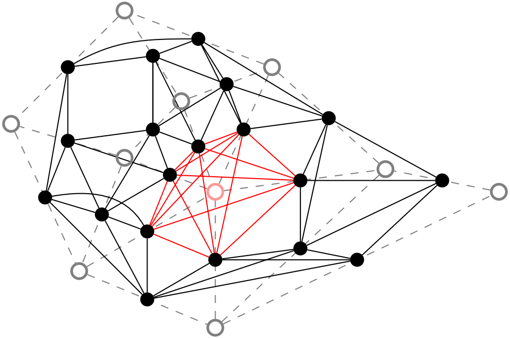
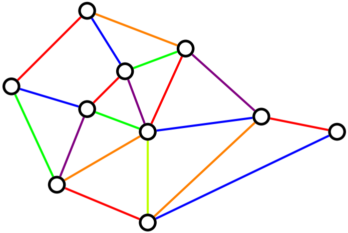


Proposition

$$\Delta(G) \leq \omega(L(G)) \leq \chi'(G) \leq 2\Delta(G) - 1.$$

Coloring edges - Vizing's Theorem



Theorem (Vizing, 1964)

If G is simple, then at most $\Delta(G) + 1$ colors are needed.

Coloring edges - chalenge

Theorem (Holyer, 1981)

Deciding whether $\Delta(G)$ or $\Delta(G)+1$ is NP-complete, even if G is cubic.

Coloring edges - chalenge

GRAPH CLASS	MEMBER		INDSET		CLIQUE		CLIPAR		CHRNUM		CHRIND		HAMCIR		DOMSET		MAXCUT		STTREE		GRAISO	
Trees/Forests	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[GJ]	P	[T]	P	[GJ]
Almost Trees (k)	P		P	[24]	P	[T]	P?		P?		P?		P?		P	[45]	P?		P?		P?	
Partial k-Trees	P	[2]	P	[1]	P	[T]	P?		P	[1]	O?		P	[3]	P	[3]	P?		P?		O?	
Bandwidth-k	P	[68]	P	[64]	P	[T]	P?		P	[64]	P?		P?		P	[64]	P	[64]	P?		P	[58]
Degree-k	P	[T]	N	[GJ]	P	[T]	N	[GJ]	N	[GJ]	N	[49]	N	[GJ]	N	[GJ]	N	[GJ]	N	[GJ]	P	[58]
Planar	P	[GJ]	N	[GJ]	P	[T]	N	[10]	N	[GJ]	O		N	[GJ]	N	[GJ]	P	[GJ]	N	[35]	P	[GJ]
Series Parallel	P	[79]	P	[75]	P	[T]	P?		P	[74]	P	[74]	P	[74]	P	[54]	P	[GJ]	P	[82]	P	[GJ]
Outerplanar	P		P	[6]	P	[T]	P	[6]	P	[67]	P	[67]	P	[T]	P	[6]	P	[GJ]	P	[81]	P	[GJ]
Halin	P		P	[6]	P	[T]	P	[6]	P	[74]	P	[74]	P	[T]	P	[6]	P	[GJ]	P?		P	[GJ]
k-Outerplanar	P		P	[6]	P	[T]	P	[6]	P	[6]	O?		P	[6]	P	[6]	P	[GJ]	P?		P	[GJ]
Grid	P		P	[GJ]	P	[T]	P	[GJ]	P	[T]	P	[GJ]	N	[51]	N	[55]	P	[T]	N	[35]	P	[GJ]
$K_{3,3}$-Free	P	[4]	N	[GJ]	P	[T]	N	[10]	N	[GJ]	O?		N	[GJ]	N	[GJ]	P	[5]	N	[GJ]	O?	
Thickness-k	N	[60]	N	[GJ]	P	[T]	N	[10]	N	[GJ]	N	[49]	N	[GJ]	N	[GJ]	N	[7]	N	[GJ]	O?	
Genus-k	P	[34]	N	[GJ]	P	[T]	N	[10]	N	[GJ]	O?		N	[GJ]	N	[GJ]	O?		N	[GJ]	P	[61]
Perfect	O!		P	[42]	P	[42]	P	[42]	P	[42]	O?		N	[1]	N	[14]	O?		N	[GJ]	I	[GJ]
Chordal	P	[76]	P	[40]	P	[40]	P	[40]	P	[40]	O?		N	[22]	N	[14]	O?		N	[83]	I	[GJ]
Split	P	[40]	P	[40]	P	[40]	P	[40]	P	[40]	O?		N	[22]	N	[19]	O?		N	[83]	I	[15]
Strongly Chordal	P	[31]	P	[40]	P	[40]	P	[40]	P	[40]	O?		O?		P	[32]	O?		P	[83]	O?	
Comparability	P	[40]	P	[40]	P	[40]	P	[40]	P	[40]	O?		N	[1]	N	[28]	O?		N	[GJ]	I	[GJ]
Bipartite	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]	P	[GJ]	N	[1]	N	[28]	P	[T]	N	[GJ]	I	[GJ]
Permutation	P	[40]	P	[40]	P	[40]	P	[40]	P	[40]	O?		O		P	[33]	O?		P	[23]	P	[21]
Cographs	P	[T]	P	[40]	P	[40]	P	[40]	P	[40]	O?		P	[25]	P	[33]	O?		P	[23]	P	[25]
Undirected Path	P	[39]	P	[40]	P	[40]	P	[40]	P	[40]	O?		O?		N	[16]	O?		O?		I	[GJ]
Directed Path	P	[38]	P	[40]	P	[40]	P	[40]	P	[40]	O?		O?		P	[16]	O?		P	[83]	O?	
Interval	P	[17]	P	[44]	P	[44]	P	[44]	P	[44]	O?		P	[53]	P	[16]	O?		P	[83]	P	[57]
Circular Arc	P	[78]	P	[44]	P	[50]	P	[44]	N	[36]	O?		O?		P	[13]	O?		P	[83]	O?	
Circle	P	[71]	P	[GJ]	P	[50]	O?		N	[36]	O?		P	[12]	O?		O?		P	[70]	O?	
Proper Circ. Arc	P	[77]	P	[44]	P	[50]	P	[44]	P	[66]	O?		P	[12]	P	[13]	O?		P	[83]	O?	
Edge (or Line)	P	[47]	P	[GJ]	P	[T]	N	[GJ]	N	[49]	O?		N	[11]	N	[GJ]	O?		N	[70]	I	[15]
Claw-Free	P	[T]	P	[63]	O?		N	[GJ]	N	[49]	O?		N	[11]	N	[GJ]	O?		N	[70]	I	[15]

Johnson. The NP-completeness column: an ongoing guide. J. of Algorithms 6 (3) (1985) 434–451.

Coloring edges - chalenge

Graph Class	Member	IndSet	Clique	CliPar	ChrNum	ChrInd	HamCir	DomSet	MaxCut	STTree	GraphIso
Trees/Forests	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [GJ]	P [T]	P [GJ]
Almost Trees (k)	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
Partial k-Trees	P [OG]	P [5]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
Bandwidth-k	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [OG]	P [76]	P [OG]
Degree-k	P [T]	N [GJ]	P [T]	N [29]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [GJ]	N [GJ]	P [OG]
Planar	P [GJ]	N [GJ]	P [T]	N [78]	N [GJ]	O	N [GJ]	N [GJ]	P [GJ]	N [OG]	P [GJ]
Series Parallel	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [OG]	P [GJ]	P [OG]	P [GJ]
Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [OG]	P [OG]	P [T]	P [OG]	P [GJ]	P [OG]	P [GJ]
Halin	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [T]	P [OG]	P [GJ]	P [118]	P [GJ]
k-Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [OG]	P [OG]	P [GJ]	P [76]	P [GJ]
Grid	P [OG]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [32]	P [T]	N [OG]	P [GJ]
$K_{3,3}$-Free*	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	P [OG]	N [GJ]	P [40]
Thickness-k	N [OG]	N [GJ]	P [T]	N [78]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [119]	N [GJ]	I [RJ]
Genus-k	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	O?	N [GJ]	P [OG]
Perfect	P [34]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [OG]	N [20]	N [GJ]	I [84]
Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [84]
Split	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [108]
Strongly Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	P [OG]	N [109]	P [OG]	I [111]
Comparability	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [94]	N [102]	N [GJ]	I [22]
Bipartite	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [94]	P [T]	N [GJ]	I [22]
Permutation	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [44]	P [OG]	N [120]	P [OG]	P [OG]
Cographs	P [T]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	P [20]	P [OG]	P [OG]
Undirected Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [13]	N [OG]	N [20]	N [RJ]	I [22]
Directed Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [99]	P [OG]	N [1]	P [OG]	P [7]
Interval	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	N [1]	P [OG]	P [OG]
Circular Arc	P [OG]	P [OG]	P [OG]	P [OG]	N [OG]	O?	P [106]	P [OG]	N [1]	P [11]	P [80]
Circle	P [OG]	P [GJ]	P [OG]	N [73]	N [OG]	O?	*N [39]*	N [71]	N [26]	P [OG]	P [68]
Proper Circ. Arc	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	O?	P [11]	P [82]
Edge (or Line)	P [OG]	P [GJ]	P [T]	N [95]	N [OG]	N [28]	N [OG]	N [GJ]	P [59]	N [19]	I [OG]
Claw-Free	P [T]	P [OG]	N [103]	N [85]	N [OG]	N [28]	N [OG]	N [GJ]	N [20]	N [19]	I [OG]

Figueiredo, Melo, Sasaki, S.. Revising Johnson's Table for the 21st century. DAM 323 (2022), 184–200.
Updated version: https://cos.ufrj.br/~celina/ftp/j/RJ-current.pdf.

Concluding remarks

Summing up

1. Upper bounds for $\chi(G)$:

   $\Delta(G) + 1$, Brook's Theorem and degeneracy;

2. Lower bounds for $\chi(G)$:

   $\frac{|V(G)|}{\theta(G)}$ and $\omega(G)$;

3. Construction of graph G s.t. $\omega(G) = 2$ and $\chi(G)$ is arbitrarily large;
4. Coloring planar graphs with 6 and 5 colors;
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So much more

- Study of many other graph classes;
- Refined computational complexity: approximation and parameterization;
- Theoretical studies gave rise to many new techniques and fields, e.g., discharging method, chromatic polynomials (and generating functions), probabilistic methods, extremal graph theory, etc;
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Muito obrigada!

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