Dynamical interaction of plant communities and conservation strategies

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San Ángel Pedregal Ecological Reserve (REPSA)

* Protected area in the central campus of the National Autonomous University of Mexico (UNAM), Mexico City.

* Urban reserve, 237.33 ha.

* Micro-environments (soil, humidity and temperature) with their own biodiversity.

* Scientific research and dissemination.

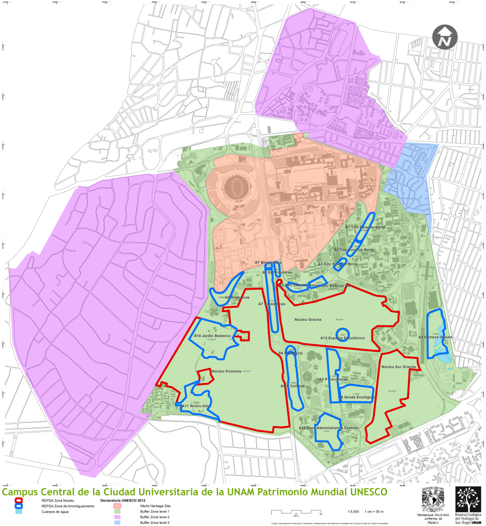
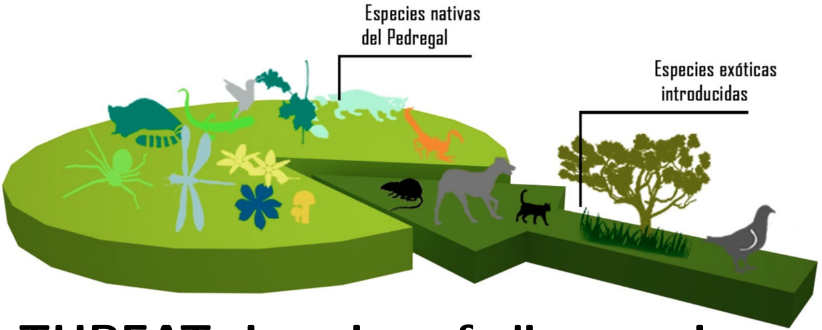


Image: REPSA, UNAM.

San Ángel Pedregal Ecological Reserve (REPSA)

"Ecosystem engineers": strong impact on plant community structure



THREAT: Invasion of alien species

* Competition for nutrients, light, water, space, etc.
* Allelopathic interference.
* Dominance of invasive species → Transition to communities with little relation to the original ones.
* Catastrophic changes in the successional trajectory.
* Effects of introducing a new species into a community?

Image: REPSA, UNAM.

Mathematical modelling

*** Focal species that govern the successional process: tepozán, palo loco, eucaliptus.**

*** Dynamic representation of key ecological interactions.**

*** Empirical observations using (limited) data from different sources.**

*** Deterministic and spatially homogeneous distribution.**

*** Resources and space are limited and constant.**

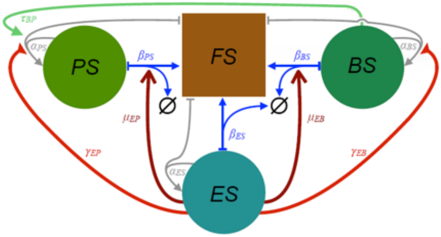
*** Variables $\sim$ trees.**

*** Units: area projected by the canopy to the soil (ha).**

[J. Acosta-Arreola, E. Domínguez-Hüttinger, P. Aguirre, N. González, J. A. Meave, *Predicting dynamic trajectories of a protected plant community under contrasting conservation regimes: insights from data-based modelling.* **(submitted)]**

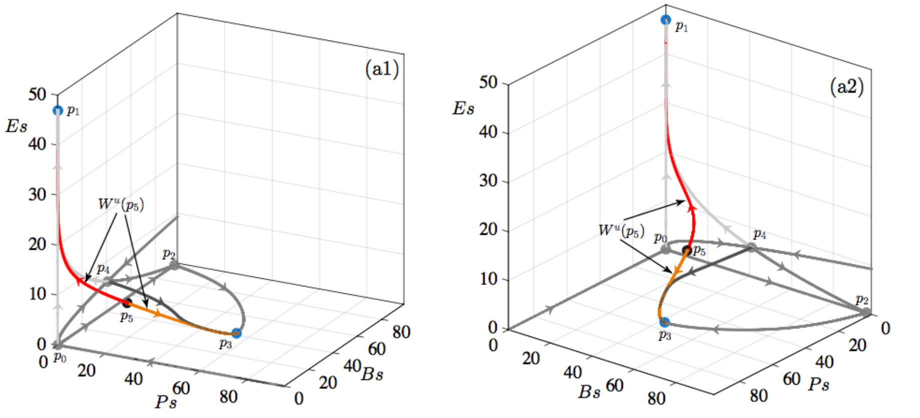
The model

$$
\begin{aligned}
\frac{dP_s}{dt} &= \frac{\alpha_{P_s} P_s F_s (1 + \tau_{B_P} B_s)}{1 + \gamma_{EP} E_s} - P_s(\beta_{P_s} + \mu_{EP} E_s), && \text{P. praecox (Palo loco)}\\
\frac{dB_s}{dt} &= \frac{\alpha_{B_s} B_s F_s}{1 + \gamma_{EB} E_s} - B_s(\beta_{B_s} + \mu_{EB} E_s), && \text{B. cordata (Tepozán)}\\
\frac{dE_s}{dt} &= \alpha_{E_s} E_s F_s - \beta_{E_s} E_s, && \text{E. camaldulensis (Eucalipto)}\\
F_s(t) &= TS - (P_s(t) + B_s(t) + E_s(t)). && \text{Espacio libre (conservation eq.)}
\end{aligned}
$$



* Positive interaction between P_s y B_s.
* E_s has negative effect on both P_s and B_s.
* α_{x_i}: reproductive rates;
* β_{x_i}: death rates $\rightarrow$ free up space;
* γ_{x_i}: Inhibition of growth by E_s;
* τ_{x_i}: Stimulation of growth;
* μ_{x_i}: Stimulation of death;
* TS: Total space

State space (P_s, B_s, E_s)



*** Multistability:**

p_1 (E_s survives, extinction of P_s and B_s)

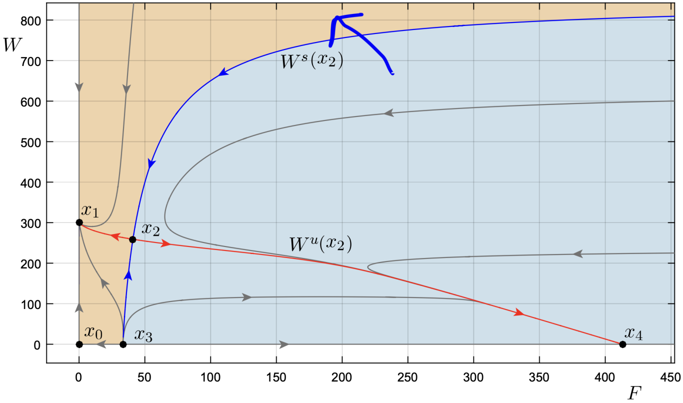
p_3 (P_s and B_s survive, E_s dies out)

p_0, p_2, p_4, p_5: saddle points (unstable).

How to decide the asymptotic behavior ($t \to \infty$) if the state at $t = 0$ is not known with certainty?

Bistability: basins of attraction and separatrices (2D toy example)

Basin of attraction: set of all combinations of (F, W)-values/concentrations/etc which, allow longterm convergence to a given equilibrium.



[**D. Contreras-Julio, PA, J. Mujica & O. Vasilieva**, *Finding strategies to regulate propagation and containment of dengue via invariant manifold analysis*, SIAM J. Appl. Dyn. Syst., 19 (2020), pp. 1392–1437.]

Stable manifold theorem

$\mathbf{x}_0$ hyperbolic equilibrium of $\boxed{\dot{\mathbf{x}} = f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n:}$

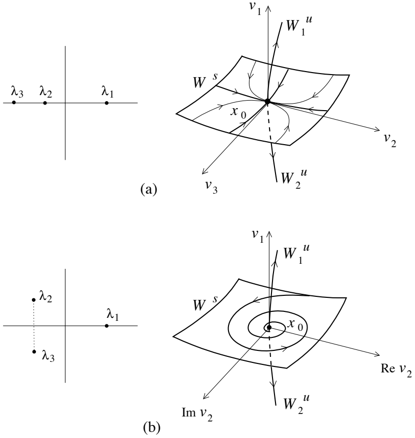
-$f(\mathbf{x}_0) = 0$

-$Df(\mathbf{x}_0)$: no eigenvalues with $\mathrm{Re}\lambda = 0$.

$W^s(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n | \, \Phi^t(\mathbf{x}) \longrightarrow \mathbf{x}_0 \text{ as } t \longrightarrow \infty\}$,

$W^u(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n | \, \Phi^t(\mathbf{x}) \longrightarrow \mathbf{x}_0 \text{ as } t \longrightarrow -\infty\}$.

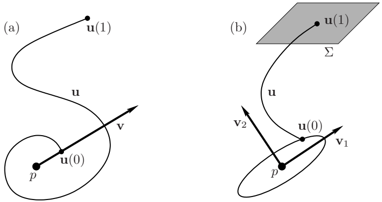
- $W^{s,u}(\mathbf{x}_0)$ are immersed manifolds in $\mathbb{R}^n$ and invariant under Φ^t.
- $\dim W^s(\mathbf{x}_0) = \dim E^s(\mathbf{x}_0)$,
  $\dim W^u(\mathbf{x}_0) = \dim E^u(\mathbf{x}_0)$.
- $T_{\mathbf{x}_0} W^s(\mathbf{x}_0) = E^s(\mathbf{x}_0)$,
  $T_{\mathbf{x}_0} W^u(\mathbf{x}_0) = E^u(\mathbf{x}_0)$.



* No analytic formulas for **global** manifolds
* Numerical methods

[**Guckenheimer & Holmes**, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, 1985.]

Computation of 2D global invariant manifolds of an hyperbolic equilibrium



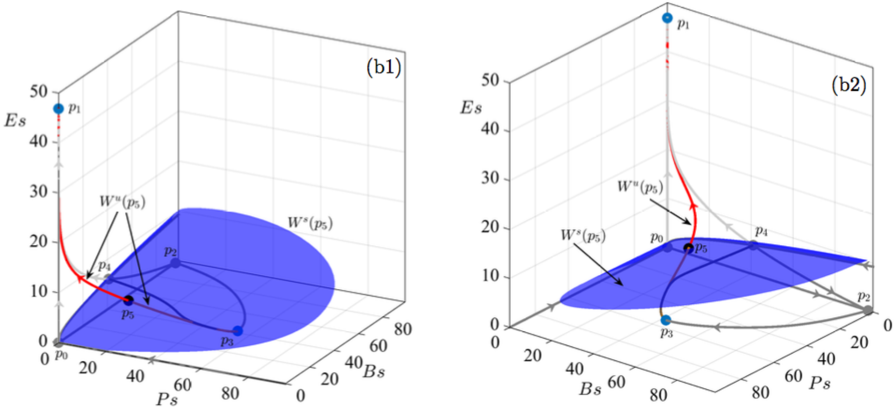
- A 2D global manifold as a solution family of BVPs (orbit segments).
- General setting:
  - $\boxed{\dot{\mathbf{u}}(t) - Tf(\mathbf{u}(t)) = \mathbf{0},}$ $\quad \mathbf{u}, f(\cdot) \in \mathbb{R}^n.$

    $\boxed{\mathbf{u}(0) \in \Omega \subset E^s(p), \qquad |\mathbf{u}(0) - p| = \delta.}$ $\quad$ Stable Manifold Theorem: Error $\sim O(\delta^2)$.

    $\boxed{\mathbf{u}(1) \in \Sigma.}$
  - Fixed integration time $T < 0$ ($\mathbf{u}(1)$ free).
  - Fixed arclength (integral condition).
- Method independent of system undergoing a bifurcation or not.
- Implementation in AUTO: Step size measures change of the *entire computed orbit segment* in state variables $\times$ parameters space. Highly accurate and fast.
- Also possible to obtain 2D manifolds of periodic orbits.

[**Krauskopf & Osinga**, *Computing invariant manifolds via the continuation of orbit segments*, in Numerical Continuation Methods for Dynamical Systems, B. Krauskopf, H. M. Osinga and J. Galán-Vioque, eds., Underst. Complex Syst., Springer-Verlag, New York, 2007.]
[**Krauskopf, Osinga, Doedel, Henderson, Guckenheimer, Vladimirsky, Dellnitz & Junge**, *A survey of methods for computing (un)stable manifolds of vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **15**, 2005.]
[**PA, Doedel, Krauskopf & Osinga**, *Investigating the consequences of global bifurcations for two-dimensional manifolds of vector fields*, Discr. Cont. Dyn. Syst. **29**, 2011.]

Stable manifold $W^s(p_5)$ of p_5



* $W^s(p_5)$: **Separatrix between basins of attraction of** p_1 **y** p_3**;**
* **Solutions starting "above"** $W^s(p_5)$ **converge to** p_1**;**
* **Solutions starting "below"** $W^s(p_5)$ **converge to** p_3**.**

Basins of attraction: what if data is incomplete?

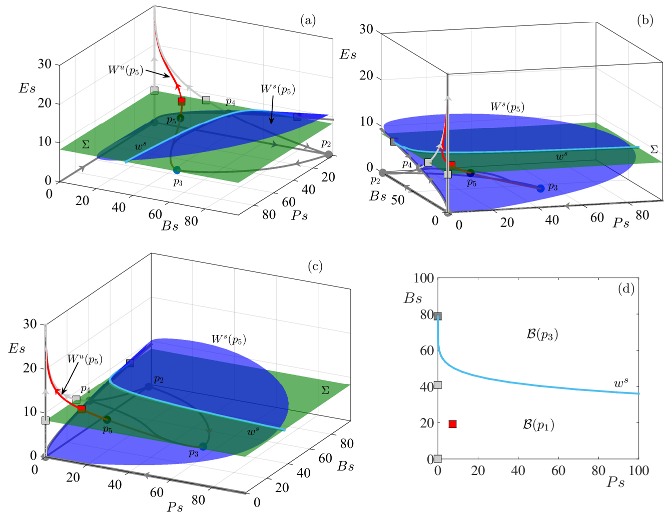
* $E_s = 8.2513$ **(most recent recorded value).**
Ps and Bs: unknown !

* **We may still say something about the possible long term outcome.**

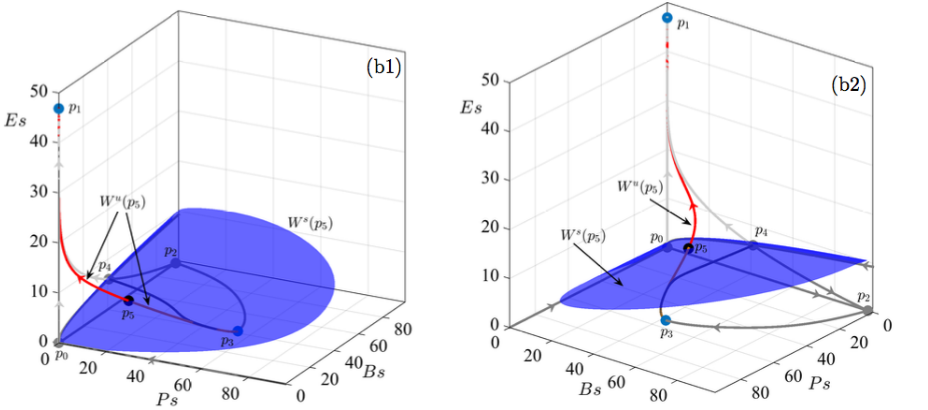
* **Panel (d): Projection of Σ onto (P_s, B_s) plane.**

* $w^s = W^s(p_5) \cap \Sigma$.

Also: Identification of regions sensitive to small variations of B_s or P_s.



Design optimal strategies to cross separatrix.



*** Human actions to eradicate E_s with minimum effort?**

1) Moving down state beyond $W^s(p_5) \to$ Cost .
2) Let the system evolve towards p_3.

Next: Two simple examples.

Strategy 1

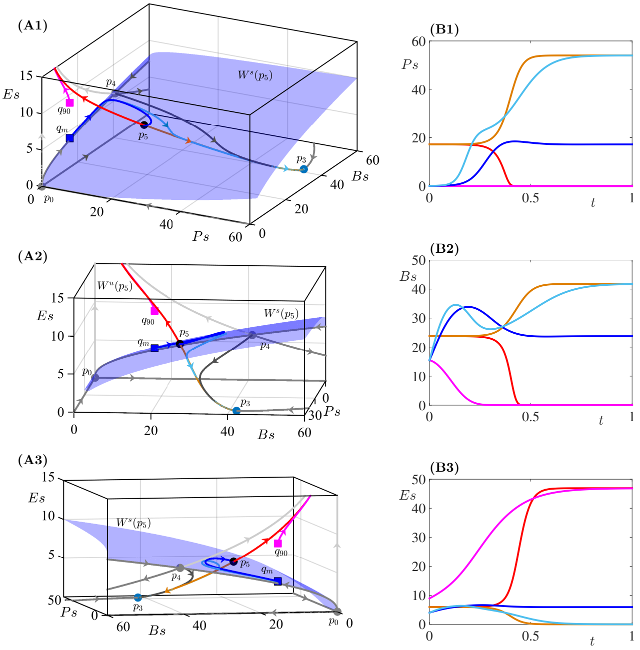
Cut down E_s.

$q_{90} \approx (0.08, 15.47, 8.94)$
State in 1990.

$q_m \approx (0.08, 15.47, 4.03)$
Nearest point to q_{90} in $W^s(p_5) \cap L$.
$L = \{P_s = 0.08,\ B_s = 15.47\}$.

Numerical continuation: Minimise distance from q_{90} to $W^s(p_5) \cap L$.

q_n: **"Just" below q_m.**



Strategy 2

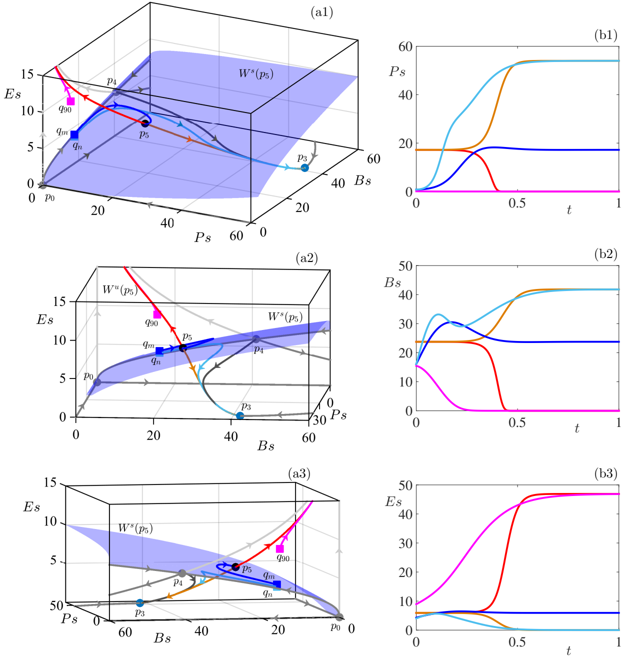
Follow shortest path in state space.

$q_{90} \approx ({\color{blue}0.08}, {\color{purple}15.47}, {\color{red}8.94})$
State in 1990.

$q_m \approx ({\color{blue}0.08}, {\color{purple}16.08}, {\color{red}4.23})$
Nearest point to q_{90} in $W^s(p_5)$.

$\rightarrow$ Cut E_s and plant B_s

q_n: "Just" below q_m.



Conclusions y perspectives

- **Model shows bistability: 2 native vs eucalyptus.**
- **Absence of interventions: Extinction of native species.**
- **Invariant manifold analysis:**
  → Size/shape/location basins of attraction.
  → Useful info to design optimal strategies to cross separatrix.
- **Idea of the best strategy for interventions: planting/cutting a particular type of tree might be much more efficient (cost-effective) than planting/cutting another one!**
- **Simple assumptions, tractable 3D model: allows analytical description, manages to capture observed qualitative properties of interaction.**
- **Model fitted with field data.... Insufficient but informative data.**
- **Problem modeled in the REPSA but that occurs in many ecosystems!**
- **Generalizable/adaptable to other species.**
- **Help in making decisions for managing reserves.**

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