A semi-hybrid mixed finite element method for coupled Stokes-Darcy flows with H(div)-conforming velocity fields

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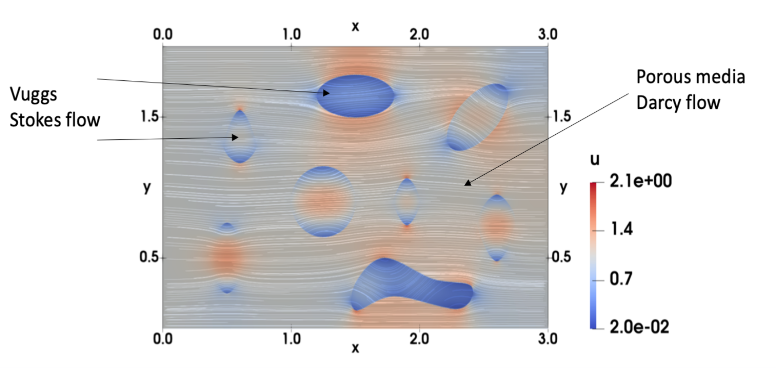
Coupled Stokes-Darcy problems: applications

Interaction between a free-fluid and a saturated porous media flow

Transport of contaminants across groundwater and surface water
Bio fluid-organ interactions
Well-reservoir coupling in petroleum engineering
Carbonate karst reservoirs (presence of vugs and caves)



New numerical model for Stokes-Darcy problems: highlights

Divergence-consistent FE pairs of spaces in $H(\mathrm{div}, \Omega) \times L^2(\Omega)$

Discretization of velocity/flux and pressure in the whole domain:
continuity of normal fluid/flux components, local mass conservation, exact divergence-free, pressure robustness[a]

Hybridization: traction Lagrange multipliers

Weakly impose tangential continuity of Stokes velocity:
natural implementation of interface Stoke-Darcy conditions, efficient solver by static condensation[b]

Computational implementation

Using tools of the FE code NeoPZ[c]:
construction of $\mathrm{H}(\mathrm{div})$-conforming spaces, hybridization and static condensation strategies are available

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[a] John, Linke, Merdon, Rebholz, SIAM Review 2017: few schemes verify these properties simultaneously

[b] Carvalho, Devloo, Gomes, <hal hal-03867520>, 2022

[c] `https://github.com/labmec/neopz`

Summary

1. Standard mixed formulation for Darcy flows in Ω[1]
2. New semi-hybrid mixed FE formulation for Stokes flows in Ω
3. Application to coupled Stokes-Dacy flows in $\Omega = \Omega_f \cup \Omega_p$
    - Combination of standard mixed formulation in Ω_p and semi-hybrid mixed formulation in Ω_f
    - Implementation aspects: static condensation
    - Some numerical studies

[1] Boff-Brezzi-Fortin, 2013

Darcy model problem: classic mixed formulation

$(\boldsymbol{u}, p)$: flux and pressure fields in the porous media $\Omega_p = \Omega$
$\mathbb{K}$: bounded symmetric positive definite (permeability) tensor, g: force
p_D, θ_N: Dirichlet and Neumann boundary data

$$\boldsymbol{u} = -\mathbb{K}\nabla p, \quad \nabla \cdot \boldsymbol{u} = g, \ \text{in} \ \Omega,$$
$$p = p_D \ \text{on} \ \Gamma_D, \quad \boldsymbol{u} \cdot \boldsymbol{n}^{\Omega} = \theta_N \ \text{on} \ \Gamma_N$$

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$$p = p_D \ \text{on} \ \Gamma_D, \quad \boldsymbol{u} \cdot \boldsymbol{n}^\Omega = \theta_N \ \text{on} \ \Gamma_N$$

$$a^{\mathcal{D}}(\boldsymbol{q}, \boldsymbol{v}) = \int_\Omega \mathbb{K}^{-1} \boldsymbol{u} \cdot \boldsymbol{q} \, dx, \quad b(\boldsymbol{u}, p) = (\nabla \cdot \boldsymbol{u}, p)$$

Find $(\boldsymbol{u}, p) \in H(\mathrm{div}; \Omega) \times L^2(\Omega)$, $\boldsymbol{u} \cdot \boldsymbol{n}^\Omega|_{\Gamma_N} = \theta_N$ verifying

$$a^{\mathcal{D}}(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = - < u_D(\mathbf{v} \cdot \boldsymbol{n}^\Omega) >_{\Gamma_D},$$

$$b(\boldsymbol{u}, \varphi) = (g, \varphi)$$

$$\forall \mathbf{v} \in H(\mathrm{div}; \Omega), \mathbf{v} \cdot \mathbf{n}^\Omega|_{\Gamma_N} = 0 \text{ and } \forall \varphi \in L^2(\Omega).$$

p: Lagrange multiplier enforcing the divergence constraint

Darcy mixed FE model

$\mathcal{T}_h$: partition of Ω; h: characteristic mesh size

flux-pressure FE pair: $\boldsymbol{X}_h \times \mathcal{Q}_h \subset H(\mathrm{div};\Omega) \times L^2(\Omega)$

$$\boldsymbol{X}_h(\theta_N) = \{\boldsymbol{v} \in \boldsymbol{X}_h; \boldsymbol{v} \cdot \boldsymbol{n}^\Omega|_{\Gamma_N} = \tilde{\theta}_N\}$$

($\tilde{\theta}_N$ L^2-projection of θ_N on $\boldsymbol{X}_h \cdot \boldsymbol{n}^\Omega|_{\Gamma_N}$)

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45 years of mixed FE discretizations (Raviart-Thomas 1977)

Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h(\theta_N) \times \mathcal{Q}_h$, verifying

$$a^{\mathcal{D}}(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = - < u_D(\mathbf{q} \cdot \boldsymbol{n}^\Omega) >_{\Gamma_D}, \quad \forall \mathbf{q} \in \boldsymbol{X}_h(0)$$

$$b(\boldsymbol{u}, \varphi) = (g, \varphi), \quad \forall \varphi \in \mathcal{Q}_h$$

Divergence-consistency : $\nabla \cdot \boldsymbol{X}_h \subset \mathcal{Q}_h$: required for stability (inf-sup)

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$$b(\boldsymbol{u}, \varphi) = (g, \varphi), \quad \forall \varphi \in \mathcal{Q}_h$$

Divergence-consistency : $\nabla \cdot \boldsymbol{X}_h \subset \mathcal{Q}_h$: required for stability (inf-sup)

Why mixed formulation?: optimal flux accuracy, locally conservative approximations, strongly divergence-free simulations, available for all standard element geometry

Remedy for the increment of DoF: hybridization + static condensation

Stokes model problem - classic mixed formulation

$(\boldsymbol{u}, p)$: velocity and pressure fields in $\Omega_f = \Omega$, $\mu > 0$: fluid viscosity; $\boldsymbol{f}$: force

Strain tensor: $D(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$; Stress tensor: $\boldsymbol{\sigma}(\boldsymbol{u}, p) = 2\mu\, D(\boldsymbol{u}) - pI$

Boundary data: $\boldsymbol{u}_D \in H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$, Solvability: $< \boldsymbol{u}_D \cdot \boldsymbol{n}^\Omega, 1 >_{\partial\Omega} = 0$

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{u} = 0 \text{ in } \quad \Omega,$$

$$\boldsymbol{u} = \boldsymbol{u}_D, \text{ on } \quad \partial\Omega$$

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$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{u} = 0 \text{ in } \quad \Omega,$$
$$\boldsymbol{u} = \boldsymbol{u}_D, \text{ on } \quad \partial\Omega$$

$$a^{\mathcal{S}}(\boldsymbol{u}, \boldsymbol{v}) = 2\mu(D(\boldsymbol{u}), D(\boldsymbol{v}))$$

Find $(\boldsymbol{u}, p) \in H^1(\Omega, \mathbb{R}^2) \times L^2_0(\Omega)$, $\boldsymbol{u}|_{\partial\Omega_D} = \boldsymbol{u}_D$, satisfying

$$a^{\mathcal{S}}(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in H^1_0(\Omega, \mathbb{R}^2),$$
$$b(\boldsymbol{u}, \varphi) = 0, \qquad \forall \varphi \in L^2_0(\Omega)$$

p: Lagrange multiplier to enforce incompressibility constraint

About 50 years of FE mixed Stokes models

Conforming methods: continuous velocity (e.g. , Taylor-Hood 1973, Mansfield 1982, MINI 1984, Girault-Raviart 1986) [2]

Non-conforming methods: weak continuity over interfaces:

- continuity on some integration points (Crouzeix-Raviart 1973)[3]
- Fully DG methods + penalization 2002
- Div-consistent FE pair + bubble velocity enrichment (2002)
- Semi-DG: Div-consistent FE pair + penalization (2009)
- Using hybridization + stabilization (HDG 2010, HHO 2014)

[2] inf-sup + efficiency not easy, Boffi-Brezzi-Fortin, Springer 2008

[3] Brenner, Forty years of the Crouzeix-Raviart element, Numer. Meth. PDE (2015)

New FE proposal for the Stokes model: (HM-H(div))

Geometry

$\mathcal{T} = \{K\}$: partition of Ω; Γ: edges of K (mesh skeleton)
$\boldsymbol{n}$: fixed vector field on Γ, $\boldsymbol{n}|_{\partial K}$ unit normal, $\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{n}^{\Omega}$

Functional spaces

$H^1(\mathcal{T}, \mathbb{R}^2)$ - broken H^1 space
$\boldsymbol{\Lambda} = \{\boldsymbol{\lambda}; \boldsymbol{\lambda}|_E = \boldsymbol{\sigma}\boldsymbol{n}|_E, E \in \Gamma; \boldsymbol{\sigma} \in H(\mathrm{div}, \Omega, \mathbb{M})\}$: tractions

$$\boldsymbol{X} = H(\mathrm{div}, \Omega) \cap H^1(\mathcal{T}, \mathbb{R}^2), \ \ \mathfrak{Q} = L^2(\Omega$$

$$a_{\mathcal{T}}(\boldsymbol{u}, \boldsymbol{v}) = 2\mu \sum_{K \in \mathcal{T}} (D(\boldsymbol{u}), D(\boldsymbol{v}))_K$$

$\boldsymbol{\Lambda}^t = \{\boldsymbol{\eta}^t; \boldsymbol{\eta} \in \boldsymbol{\Lambda}\}$, $\boldsymbol{\eta}^t = \boldsymbol{\eta} - (\boldsymbol{\eta} \cdot \boldsymbol{n})\boldsymbol{n}$ tangential components

$$\langle \boldsymbol{\eta}^t, \boldsymbol{u} \rangle_{\partial\mathcal{T}} = \sum_{K \in \mathcal{T}} \langle \boldsymbol{\eta}^t, \boldsymbol{u} \rangle_{\partial K}$$

New FE proposal for the Stokes model: (HM-H(div))

HM-H(div) [a]: Find $(\boldsymbol{u}, p, \boldsymbol{\lambda}^t, \rho) \in \boldsymbol{X} \times \mathcal{Q} \times \boldsymbol{\Lambda}^t \times \mathbb{R}$ such that

$$
\begin{aligned}
a_{\mathcal{T}}(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + \langle \boldsymbol{\lambda}^t, \boldsymbol{v} \rangle_{\partial \mathcal{T}} &= (\boldsymbol{f}, \boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{X}, \\
b(\boldsymbol{u}, q) + (\rho, q) &= 0, & \forall q \in \mathcal{Q} \\
\langle \boldsymbol{\eta}^t, \boldsymbol{u} \rangle_{\partial \mathcal{T}} &= < \boldsymbol{\eta}^t, \boldsymbol{u}_D >_{\partial\Omega} & \forall \boldsymbol{\eta}^t \in \boldsymbol{\Lambda}^t \\
(\xi, p) &= 0 & \forall \xi \in \mathbb{R}
\end{aligned}
$$

Three multipliers to enforce three constraints

p: for divergence-free; ρ: for zero-mean pressure

$\boldsymbol{\lambda}^t$: for interface continuity of tangential velocity;

[a] Inspired in the full-hybrid mixed formulation for velocities in $H^1(\mathcal{T}, \mathbb{R}^2)$ Araya, Harder, Poza, Valentin, CMAME 2017

Theorem

$(\boldsymbol{u}, p, \boldsymbol{\lambda}^t, \rho)$ solves HM-H(div) $\Longleftrightarrow$ $(\boldsymbol{u}, p)$ solves MF; Moreover, $\rho = 0$ and $\boldsymbol{\lambda}^t$ is the tangential component of $\boldsymbol{\lambda} = \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}$

FE discretizations: HM-H(div) $(\mathcal{S}_h)$

$\mathcal{T}_h$: partitions of Ω; h: characteristic mesh size

$$\mathcal{S}_h = \boldsymbol{X}_h \times \mathcal{Q}_h \times \boldsymbol{\Lambda}_h^t \subset \boldsymbol{X} \times \mathcal{Q} \times \boldsymbol{\Lambda}^t$$

Velocity-pressure FE pair: $\boldsymbol{X}_h \times \mathcal{Q}_h$ Multiplier FE space: $\boldsymbol{\Lambda}_h^t$

HM-H(div) $(\mathcal{S}_h)$: find $(\boldsymbol{u}_h, p_h, \boldsymbol{\lambda}_h^t, \rho) \in \mathcal{S}_h \times \mathbb{R}$ such that

$$\begin{aligned}
a_{\mathcal{T}_h}(\boldsymbol{u}_h, \boldsymbol{v}) - b(\boldsymbol{v}, p_h) + \langle \boldsymbol{\lambda}_h^t, \boldsymbol{v} \rangle_{\partial\mathcal{T}} &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{X}_h, \\
b(\boldsymbol{u}_h, q) + (\rho, q) &= 0, && \forall q \in \mathcal{Q}_h, \\
\langle \boldsymbol{\eta}^t, \boldsymbol{u}_h \rangle_{\partial\mathcal{T}} &= < \boldsymbol{\eta}^t, \boldsymbol{u}_D >_{\partial\Omega}, && \forall \boldsymbol{\eta}^t \in \boldsymbol{\Lambda}_h^t, \\
(\xi, p_h) &= 0, && \forall \xi \in \mathbb{R}.
\end{aligned}$$

HM-H(div) $(\mathcal{S}_h)$: Remarks on solvability

Hypotheses of Brezzi's theory for constrained variational formulations [a]:

$\boldsymbol{X}_h$-ellipticity: $||\boldsymbol{v}||_{a_{\mathcal{T}_h}} = \sqrt{a_{\mathcal{T}_h}(\boldsymbol{v}, \boldsymbol{v})}$ is a norm in $\boldsymbol{X}_h$ OK

Divergence-compatibility between $\{\boldsymbol{X}_h, Q_h\}$: $\nabla \cdot \boldsymbol{X}_h = Q_h$ OK (known cases for standard geometry)[b]

Trace compatibility between $\{\boldsymbol{X}_h, \boldsymbol{\Lambda}_h^t\}$: $\boldsymbol{\Lambda}_h^t(0) = \{0\}$ (not so easy)

$$\boldsymbol{\Lambda}_h^t(0) = \{\boldsymbol{\eta}^t \in \boldsymbol{\Lambda}_h^t; \langle \boldsymbol{\eta}^t, \boldsymbol{u} \rangle_{\partial \mathcal{T}_h} = 0, \forall \boldsymbol{u} \in \boldsymbol{Z}_h\}$$

$$\boldsymbol{Z}_h = \{\boldsymbol{v} \in \boldsymbol{X}_h; b(\boldsymbol{v}, q) = 0, \forall q \in Q_h\} \text{(divergence-free fields)}$$

[a] Brezzi, RAIRO, 1973

[b] Boffy-Brezzi-Fortin, Springer-Verlag, 2013

Theorem

Assume divergence and trace compatibility conditions are verified. Then the HM-H(div) $(\mathcal{S}_h)$ method has a unique solution.

Error estimates for the HM-H(div) ($\mathcal{S}_h$) method

Theorem

Let $(\boldsymbol{u}, p, \boldsymbol{\lambda}^t) \in \boldsymbol{X} \times \mathcal{Q} \times \boldsymbol{\Lambda}^t$ *be the exact solution of the HM-H(div). If* $(\boldsymbol{u}_h, p_h, \boldsymbol{\lambda}_h^t) \in \mathcal{S}_h$ *is a HM-H(div) ($\mathcal{S}_h$) solution, then*

$$||\boldsymbol{u} - \boldsymbol{u}_h||_{a_{\mathcal{T}_h}} \lesssim \inf_{\boldsymbol{z} \in \boldsymbol{Z}_h} ||\boldsymbol{u} - \boldsymbol{z}||_{a_{\mathcal{T}_h}} + \sup_{\boldsymbol{v} \in \boldsymbol{X}_h \setminus \{0\}} \frac{\langle \boldsymbol{\lambda}_h^t - \boldsymbol{\lambda}^t, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h}}{||\boldsymbol{v}||_{a_{\mathcal{T}_h}}} (\textit{consistency} \quad \textit{error})$$

$$||p - p_h||_{L^2} \lesssim \inf_{q \in \mathcal{Q}_{h0}} ||p - q||_{L^2} + \sqrt{\mu} \inf_{\boldsymbol{z} \in \boldsymbol{Z}_h} ||\boldsymbol{u} - \boldsymbol{z}||_{a_{\mathcal{T}_h}} + \sup_{\boldsymbol{v} \in \boldsymbol{X}_h \setminus \{0\}} \frac{\langle \boldsymbol{\lambda}_h^t - \boldsymbol{\lambda}^t, \boldsymbol{v} \rangle_{\partial \mathcal{T}}}{||\boldsymbol{v}||_{1, \mathcal{T}_h}}$$

Proof: similar to usual analysis of non-conforming methods[a]

[a] Tai, Whither, Calcolo (2006)

Implemented examples in 2D

$\boldsymbol{X}_h \times \mathcal{Q}_h$: Local FE spaces $\boldsymbol{X}_k(K) \times \Psi_k(K)$, $K \in \mathcal{T}_h$
backtracted from $\boldsymbol{X}_k(\hat{K}) \times \Psi_k(\hat{K})$
k: degree of normal traces of $\boldsymbol{v} \in \boldsymbol{X}_k(\hat{K})$ on $\partial\hat{K}$

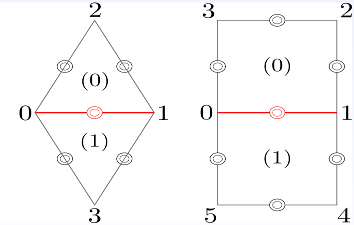
$\hat{K}$	Method	$\boldsymbol{X}_k(\hat{K})$	$\Psi_k(\hat{K})$
$\hat{T}$	$BDM(k)$	$\mathbb{P}_k(\hat{K}, \mathbb{R}^2)$	$\mathbb{P}_{k-1}(\hat{K})$
	$BDM^+(k)$	$\mathbb{P}^\partial_k(\hat{K}, \mathbb{R}^2) \oplus \mathring{\mathbb{P}}_{k+1}(\hat{K}, \mathbb{R}^2)$	$\mathbb{P}_k(\hat{K})$
$\hat{R}$	$RT([k])$	$\mathbb{Q}_{k+1,k} \times \mathbb{Q}_{k,k+1}(\hat{K})$	$\mathbb{Q}_{k,k}(\hat{K})$

$\boldsymbol{\Lambda}^t_h$: Local FE spaces $\boldsymbol{W}(E)$, $E \subset \Gamma$: backtracted from $\mathbb{P}_{k-1}(\hat{E})\boldsymbol{\tau}^{\hat{E}}$

Solvability: numerical rank test

Traction compatibility $\mathbf{\Lambda}_h^t(0) = \{\boldsymbol{\mu} \in \mathbf{\Lambda}_h^t ; \langle \boldsymbol{\eta}^t, \boldsymbol{u} \rangle_{\partial\mathcal{T}} = 0, \forall \boldsymbol{u} \in \boldsymbol{Z}_h\} = \{0\}$
as a rank condition: $\mathrm{rank}(\mathcal{M}) = \dim \mathbf{\Lambda}^t$
$\mathcal{M} = [\langle \boldsymbol{\eta}_i^t, \boldsymbol{v}_j \rangle_{\partial\mathcal{T}}]$ for bases $\{\boldsymbol{\eta}_i^t\}$ of $\mathbf{\Lambda}_h^t$ and $\{\boldsymbol{v}_j\}$ of $\boldsymbol{Z}_h$

Partitions: two triangular or two quadrilateral elements K_i, $i = 0, 1$



	Rank of $\mathcal{M}$		$\dim \tilde{\mathbf{\Lambda}}^t$
k	$BDM(k)$	$BDM^+(k)$	
1	4	5	5
2	9	10	10
3	14	15	15

	Rank of $\mathcal{M}$		$\dim \tilde{\mathbf{\Lambda}}^t$
k	$RT(k)$	$RT^+(k)$	
1	7	7	7
2	14	14	14
3	21	21	21

$RT(k)$, $RT^+(k)$ and $BDM^+(k)$ passed the rank test

Numerical verification tests [1]

$$\Omega = (0,2) \times (-1,1), \; p = \cos x \sin y - p_0 \in L^2_0(\Omega)$$

$$\boldsymbol{u} \;=\; \nabla \times \psi, \quad \text{where} \quad \psi = -\sin x \cos y,$$

Uniform conformal partitions $\mathcal{T}_h = \{K\}$ of triangular or quadrilateral K,
$h = 2/N, N = 2^i, i = 2, \cdots 6;$
$BDM^+(k)$ or $RT(k)$

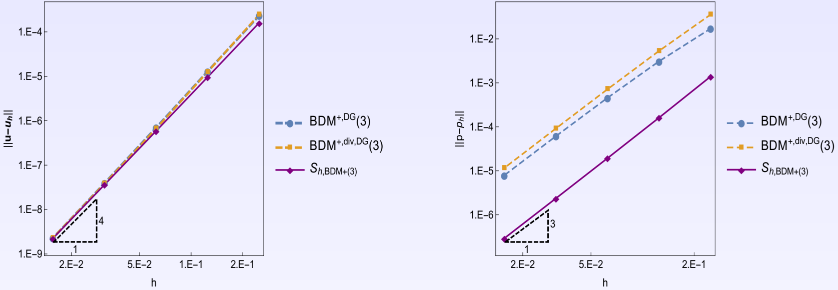
Comparisons with SIP-DG methods [2]

[1] Botti, Di Pietro, Droniou, A hybrid high-order discretisation of the Brinkman problem robust in the Darcy and Stokes limits, CMAME (2018)

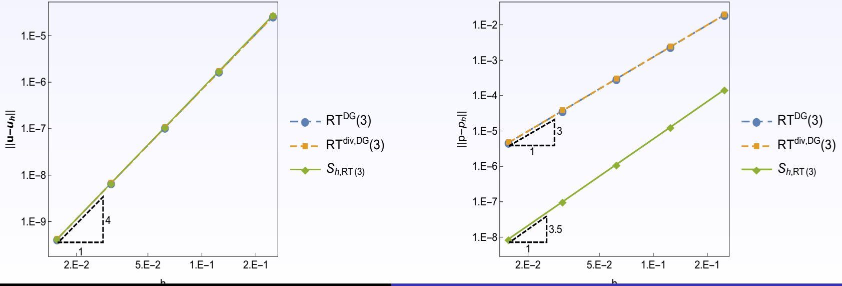
[2] Carvalho,Devloo, Gomes, On the use of divergence balanced H(div)-L2 pair of approximation spaces for divergence-free and robust simulations of Stokes, coupled Stokes–Darcy and Brinkman problems , MATCOMP (2020) (survey)

Stokes problem: HM-H(div)($\tilde{S}$) x SIP-DG

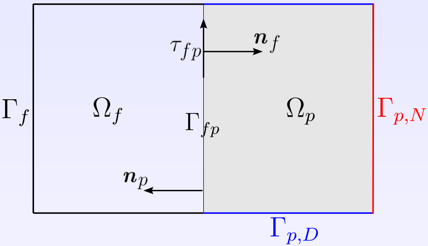
In triangles



In quadrilaterals



Coupled Stokes-Darcy problem: boundary conditions



At Γ_{fp}: (Beavers–Joseph–Saffman)

$$\boldsymbol{u}_p \cdot \vec{n}^f + \boldsymbol{u}_f \cdot \vec{n}^p = 0; \text{flux continuity}$$

$$-2\mu[D(\boldsymbol{u}_f)\boldsymbol{n}_f] \cdot \boldsymbol{n}_f + p_f = p_p; \text{balance of normal forces}$$

$$-2[D(\boldsymbol{u}_f)\boldsymbol{n}_f] \cdot \boldsymbol{t}_{fd} = \alpha_{BJS} \boldsymbol{u}_f \cdot \boldsymbol{t}_{fd} \text{friction condition}$$

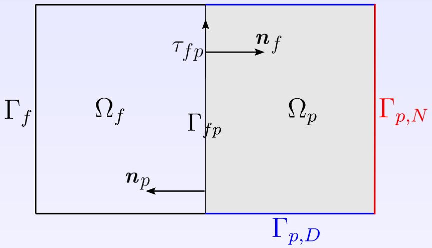
At $\partial\Omega$

$$\boldsymbol{u}_f = \boldsymbol{u}_D, \text{ on } \Gamma_f$$

$$\boldsymbol{u}_p \cdot \boldsymbol{n}^\Omega = \theta_N \text{ on } \Gamma_{p,N}$$

$$p = p_D \quad \text{on } \Gamma_{p,D} (\neq \emptyset)$$

Semi-hybrid mixed FE settings for Stokes-Darcy problems



Geometry: (index p for flow in porous and f for free flow)

Conformal partitions: $\mathcal{T}_f$ for Ω_f and $\mathcal{T}_p$ for Ω_p,

Assume that $\mathcal{T}_h = \mathcal{T}_f \cup \mathcal{T}_p$ is conformal along Γ_{fp}.

Γ_f: edges of $\mathcal{T}_f$ not included in Γ_{fp}

FE spaces:

Divergence-compatible FE pair $\boldsymbol{X}_h \times \mathcal{Q}_h$ based on $\mathcal{T}_h$

$$\boldsymbol{X}_h(\theta_N) = \{\boldsymbol{v} = (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{X}_h; \boldsymbol{v}_p \cdot \mathbf{n}_p|_{\Gamma_{pN}} = \theta_N\}$$

Multiplier FE space $\boldsymbol{\Lambda}_h^t$: based on $\Gamma_f \cup \Gamma_{fp}$

$\{\boldsymbol{X}_h|_{\Omega_f}, \boldsymbol{\Lambda}_h^t\}$: verifying the trace compatibility condition in Ω_f

Couple Stokes-Darcy problem)

$$a_h^{fp}(\boldsymbol{u},\boldsymbol{v}) := 2\mu \sum_{K\in\mathcal{T}_f} (D(\boldsymbol{u}), D(\boldsymbol{v}))_K + \sum_{K\in\mathcal{T}_p} (\mathbb{K}^{-1}\boldsymbol{u},\boldsymbol{v})_K.$$

$$\mathcal{S}_h^{fp}(\theta_N) = \boldsymbol{X}_h(\theta_N) \times \mathcal{Q}_h \times \boldsymbol{\Lambda}_h^t$$

HM-H(div)($\mathcal{S}_h^{fp}(\theta_N)$): find $(\boldsymbol{u}_h, p_h, \boldsymbol{\lambda}_h^t) \in \mathcal{S}_h^{fp}(\theta_N)$ satisfying

$$\begin{aligned} a_h^{fp}(\boldsymbol{u}_h,\boldsymbol{v}) + (\boldsymbol{v}, p_h) + \langle \boldsymbol{\lambda}_h^t, \boldsymbol{v}\rangle_{\Gamma_f\cup\Gamma_{fp}} &= \langle \boldsymbol{v}\cdot\boldsymbol{n}_p, p_D\rangle_{\Gamma_{p,D}} \\ &\quad + (\boldsymbol{f},\boldsymbol{v})_{\Omega_f}, \ \forall \boldsymbol{v}\in\boldsymbol{X}_h(0) \\ b(\boldsymbol{u}_h, q) &= -(g,q)_{\Omega_p}, \ \forall q\in\mathcal{Q}_h \\ \langle \boldsymbol{\eta}^t, \boldsymbol{u}_h\rangle_{\Gamma_f\cup\Gamma_{fp}} - \alpha_{BJS}^{-1}\langle \boldsymbol{\eta}^t, \boldsymbol{\lambda}_h^t\rangle_{\Gamma_{fp}} &= \langle \boldsymbol{\eta}^t, \boldsymbol{u}_D\rangle_{\Gamma_f}, \ \forall \boldsymbol{\eta}^t\in\boldsymbol{\Lambda}_h^t \end{aligned}$$

Theorem

The HM-H(div) ($\mathcal{S}_h^{fp}$) method has a unique solution.

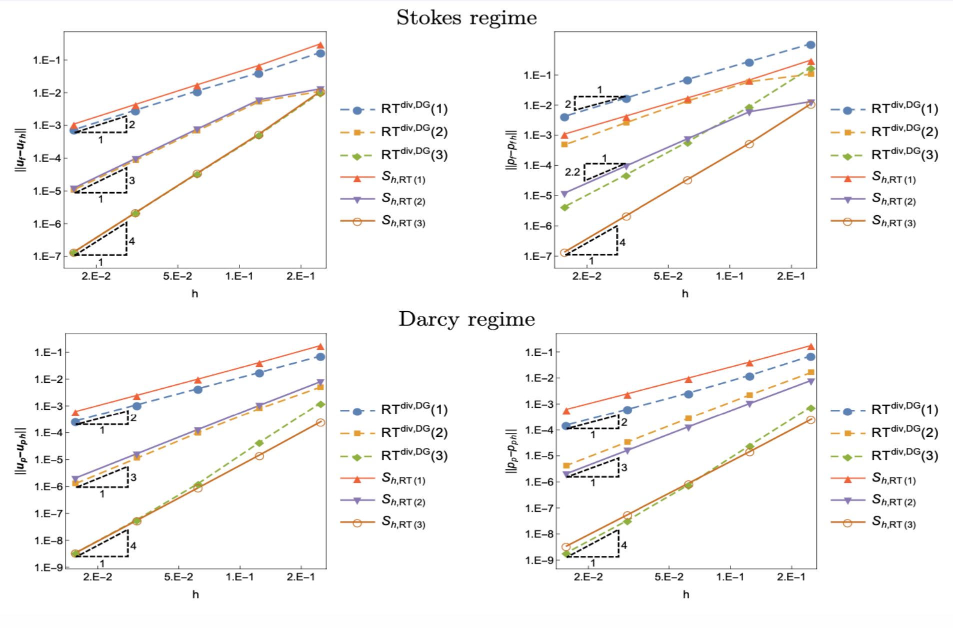
Stokes-Darcy: test problem

$$\Omega_f = (0,\pi) \times (0,1) \qquad \Omega_p = (0,\pi,\pi) \times (-1,0)$$
$$\Gamma_{fp} = \{0 < x < \pi; y = 0\} \quad \alpha_{BJS} = 1$$

$$\mathbf{u}_f = \begin{bmatrix} v'(y)\cos x \\ v(y)\sin x \end{bmatrix}, \quad p_f = \sin x \sin y, \quad \text{where} \quad v(y) = \frac{1}{\pi^2}\sin^2(\pi y) - 2,$$
$$\mathbf{u}_p = \begin{bmatrix} (e^{-y} - e^{y})\cos x \\ -(e^{-y} + e^{y})\sin x \end{bmatrix}, \quad p_p = (-e^{-y} + e^{y})\sin x.$$

Forcing functions and boundary data: taken from the exact solution

Stokes-Darcy: verification and comparison results



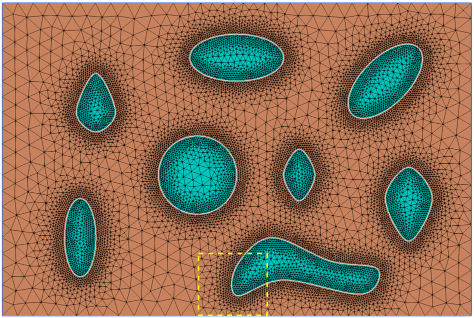
Stokes-Darcy: multiple vugs

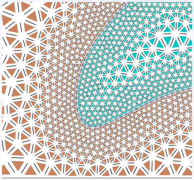
$\Omega = (0,3) \times 0,2)$ Ω_f: vugs; Ω_p: surrounding the vugs; $\Gamma_{fp} = \partial\Omega_f$

$\mu = 0.1$, $\alpha_{BJS} = 1$ $\mathbb{K} = 0.5\mathbb{I}$ in Ω_p, $g = 0$ and $\mathbf{f} = \mathbf{0}$

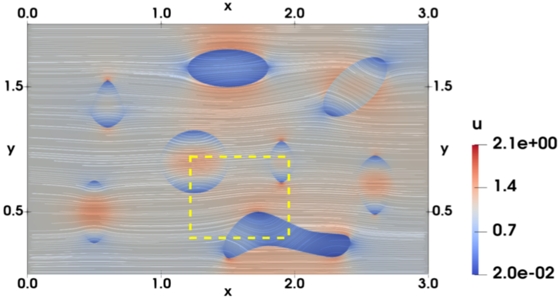
DC $p_D = 0$: right side NC : $\theta_N = 0$ bottom + top sides, $\theta_N = 1$ left side

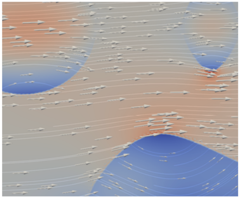
Unstructured triangular mesh: refinement $\approx \Gamma_{fp}$

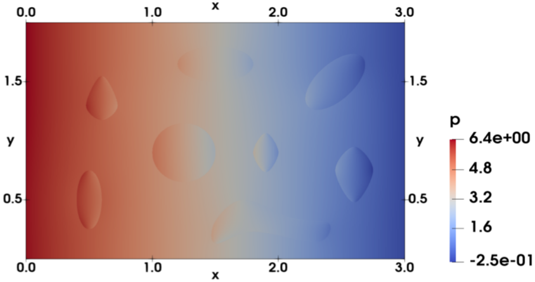




Stokes-Darcy: numerical solution







Comments on implementation: static condensation

$\mathbf{X}_h = \mathbf{X}_h^{\partial} \oplus \mathring{\mathbf{X}}_h$ (trace type $\oplus$ internal type)

$\mathcal{Q}_h = \mathcal{Q}_{0,h} \oplus \mathcal{Q}_h^{\perp}$ (piecewise constants $\oplus$ piecewise zero − mean)

$$\boldsymbol{u}_h = \boldsymbol{u}_h^{\partial} + \mathring{\boldsymbol{u}} \qquad p_h = \bar{u}_h + u_h^{\perp}$$

$\delta_h \in \mathcal{Q}_{0,h}$: new multiplier to enforce the constraint $p_h - \bar{p}_h \in \mathcal{Q}_h^{\perp}$

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Primary DoF $\boldsymbol{V}_1 = (\hat{\boldsymbol{u}}_1, \hat{p}_1, \hat{\boldsymbol{\lambda}})^T$: for $(\boldsymbol{u}_h^{\partial}, \bar{p}_h, \boldsymbol{\lambda}_h^t)$
Secondary DoF $\boldsymbol{V}_0 = (\hat{\boldsymbol{u}}_0, \hat{p}_0, \hat{\delta})^T$: for $(\mathring{\boldsymbol{u}}_h, p_h, \delta_h)$

Local matrix structure in $K \in \mathcal{T}_h$

$$\left[\begin{array}{c|c} \boldsymbol{K}_{00} & \boldsymbol{K}_{01} \\ \hline \boldsymbol{K}_{10} & \boldsymbol{K}_{11} \end{array}\right] \cdot \left[\begin{array}{c} \boldsymbol{V}_0 \\ \hline \boldsymbol{V}_1 \end{array}\right] = \left[\begin{array}{c} \boldsymbol{f}_0 \\ \hline \boldsymbol{f}_1 \end{array}\right]$$

Global matrix structure $\boldsymbol{K}$ and $\boldsymbol{f}$

assembly of $\boldsymbol{K}_{11} - \boldsymbol{K}_{10}\, \boldsymbol{K}_{00}^{-1}\, \boldsymbol{K}_{01}$ and $\boldsymbol{f}_1 - \boldsymbol{K}_{01}\, \boldsymbol{K}_{00}^{-1} \cdot \boldsymbol{f}_0$

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Algorithm

1. **Compute** $\boldsymbol{V}_1$ **by solving the global system**: $\boldsymbol{K}\boldsymbol{V}_1 = \boldsymbol{f}$
2. **Recover** $\boldsymbol{V}_0$: $\boldsymbol{K}_{00}\boldsymbol{V}_0 = \boldsymbol{f}_0 - \boldsymbol{K}_{01}\boldsymbol{V}_1$ (by independent local solvers)
3. **Post-process** the solution from the DoF

Final comments: highlights

Semi-hybrid mixed method for Stokes-Darcy problems using divergence-compatible FE pairs for velocity-pressure fields; Solvability requires a tangential traction compatibility property

Implementation by static condensation: coarse DoF for primary variables solved globaly and refined details (second valriables) recovered by independent local systems

Applications for some classic divergence-compatible FE pairs: comparison with SIP DG methods show similar convergence rates; Stability and convergence require further studies under investigation;

Important properties: strongly local conservative, exact divergence-free velocity fields, pressure robustness

Applications to other nonlinear physical problems (NS) and a multiscale scheme were developed (P.G.S. Carvalho, PhD thesis, FECFAU-Unicamp, 2021)

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