Nematic liquid crystals: well posedness, optical solitons and control

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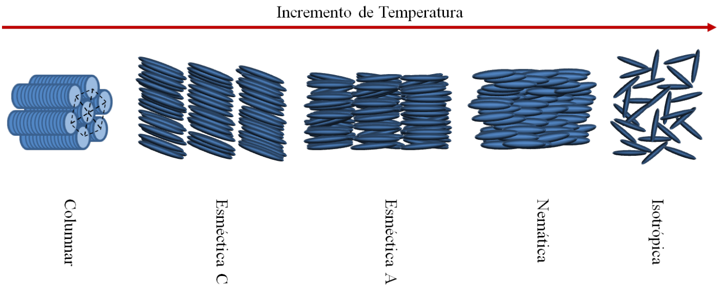
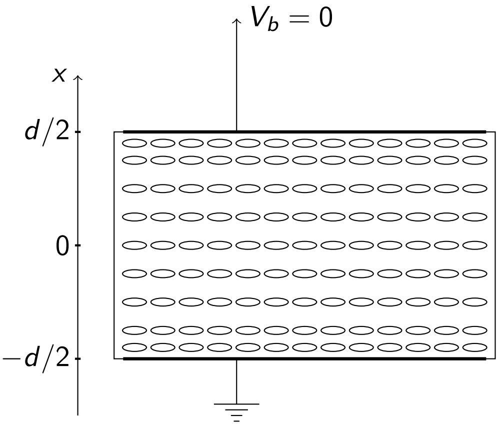
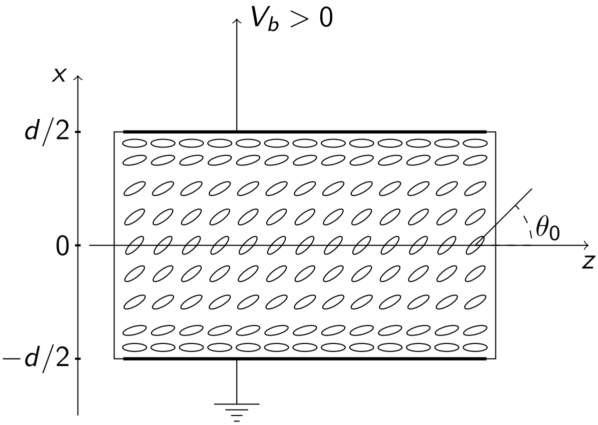
Nematic Liquid Crystals



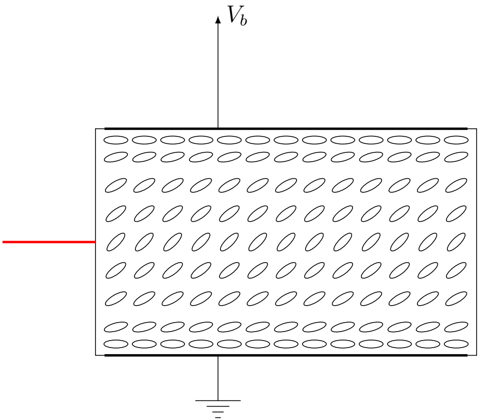
Scheme of the experimental device



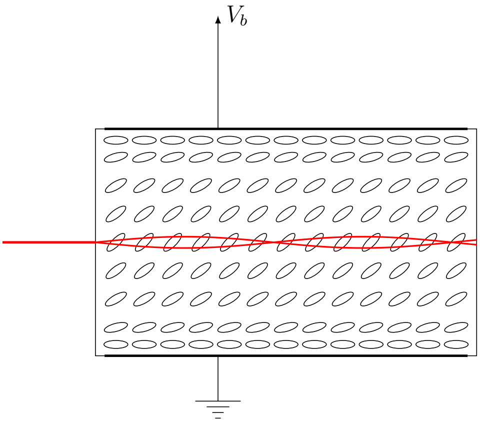
Scheme of the experimental device



Scheme of the experimental device



Scheme of the experimental device



Full Nematicon Model

Wave envelope: $u = A/E_b$ and deflection angle: $\theta = \psi - \theta_0$

$$\begin{cases} iu_z = -\frac{1}{2}\nabla_x^2 u - \gamma(\sin^2(\theta_0 + \theta) - \sin^2(\theta_0))u, \\ \nu\nabla_x^2\theta = \frac{1}{2}E_b^2\sin(2\theta_0) - \frac{1}{2}(E_b^2 + |u|^2)\sin(2(\theta_0 + \theta)). \end{cases} \quad (1)$$

Peccianti, M., De Rossi, A., Assanto, G., De Luca, A., Umeton, C. and Khoo IC. Electrically assisted self-confinement and waveguiding in planar nematic liquid crystal cells. *Applied Physics Letters*. 2000.

Pretilt nematicon model

Assuming $\theta_0 \sim \pi/4$, $|\theta| \ll 1$, $q > 0$ depending on θ_0, we have the simpler nonlinear model:

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + \sin(2\theta)u = 0, \\ \\ \nu\nabla^2\theta - q\sin(2\theta) + 2|u|^2\cos(2\theta) = 0. \end{cases} \quad (2)$$

In this model it is assumed that the bias electric field is constant.

The linear model

Panayotaros, P. and Marchant, TR. Solitary waves in nematic liquid crystals. *Physica D: Nonlinear Phenomena*. 2014.

Considering $|\theta| \ll 1$, then $\cos(2\theta) \approx 1$ and $\sin(2\theta) \approx 2\theta$.

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + 2\theta u = 0, \\ \nu\nabla^2\theta - 2q\theta + 2|u|^2 = 0, \end{cases} \tag{3}$$

where q, ν are positive constants, $x \in \mathbb{R}^2$.

Nematicon lineal model

- Equation on θ can be solved

$$\theta(u) = G(|u|^2)(x) = \frac{2}{\nu}\int_{\mathbb{R}^2} N_0(\sqrt{2q/\nu}(x-y))|u(y)|^2 dy$$

N_0: Modified Bessel function.

- Equation on u satisfies

$$iu_z + \frac{1}{2}\nabla^2 u + 2G(|u|^2)u = 0, \;\; u(0) = u_0$$

- Well posedness (global existence, uniqueness and continuous dependence):

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2)).$$

- Existence of the stationary solutions $u(x,z) = e^{-i\omega z}\phi(x)$, with $\|\phi\|_{L^2} > a_0$.

Pre-tilted nematicon model

Borgna, J.P. and Panayotaros, P. and Rial, D and Sánchez de la Vega, C. Optical solitons in nematic liquid crystals: model with saturation effects. *Nonlinearity*. 2018.

We analyze the nonlinear system (2):

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + \sin(2\theta)u = 0, \\ \\ \nu\nabla^2\theta - q\sin(2\theta) + 2|u|^2\cos(2\theta) = 0, \end{cases}$$

$u(x, z = 0) = u_0(x)$ for $x \in \mathbb{R}^2$,

assuming $\theta_0 \sim \pi/4$, $|\theta| \ll 1$, $q > 0$ depending on θ_0.

Results

- Director angle equation: We prove the existence of $\theta = \theta(u)$ and uniqueness of director angle equation.
- Replacing $\theta(u)$ into the nonlinear Schrödinger equation, we solve de following initial value problem:

$$iu_z + \frac{1}{2}\nabla^2 u + \sin(2\theta(u))u = 0, \quad u(x, z = 0) = u_0(x).$$

- Well posedness.
- Existence of stationary solutions for laser power above a threshold.
- Saturation effect: $0 \leq \theta \leq \pi/4$ and if $\|u\|_{L^\infty} \to \infty$, then

$$\theta \to \pi/4$$

Full nematicon model

Borgna, J.P. and Panayotaros, P. and Rial, D and Sánchez de la Vega, C. Optical solitons in nematic liquid crystals: arbitrary deviation angle model. *Physica D: Nonlinear Phenomena*. 2020.

$$\begin{cases} iu_z = -\frac{1}{2}\nabla^2 u - \gamma(\sin^2(\theta_0+\theta) - \sin^2(\theta_0))u, \\ \nu\nabla^2\theta = \frac{1}{2}E_b^2\sin(2\theta_0) - \frac{1}{2}(E_b^2 + |u|^2)\sin(2(\theta_0+\theta)). \end{cases}$$

$u(x, z=0) = u_0(x)$ for $x \in \mathbb{R}^2$,

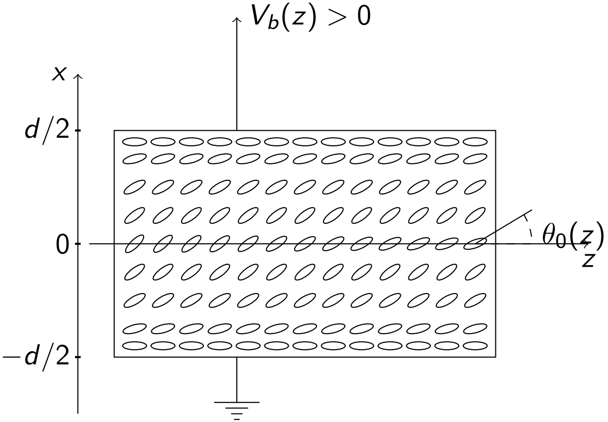
$E_b > 0$, $\theta_0 \in (\pi/4, \pi/2)$ and no assumptions on the size of θ.

Results

- Well posedness.
- Existence of stationary solutions.
- Saturation effect: $0 \leq \theta \leq \pi/2 - \theta_0$ and if $\|u\|_{L^\infty} \to \infty$, then

$$\theta + \theta_0 \to \pi/2.$$

The device



The optimal control model

We analyze the optimal control problem

$$\mathcal{J}_\star = \inf_{q \in \mathcal{Q}_{\mathrm{ad}}} \mathcal{J}(q)$$

where $\mathcal{J}(q) = \|u[q](\zeta) - u_1\|^2_{L^2} + \kappa \|\dot{q}\|^2_{L^2([0,\zeta])}$ and (q, θ, u) satisfy system (3):

$$\begin{cases} iu_z + \frac{1}{2}\nabla^2 u + 2\theta u = 0, \\ \nu \nabla^2 \theta - 2q(z)\theta + 2|u|^2 = 0, \end{cases} \qquad (4)$$

$u(x, z = 0) = u_0(x)$ for $x \in \mathbb{R}^2$ and

$\mathcal{Q}_{\mathrm{ad}} = \{q \in H^1([0,\zeta]) : q(0) = q_0 \text{ and } m \leq q(z) \leq M \text{ for all } z\}.$

Results

We proved:

- Well posedness and regularity.
- Existence of a minimum.
- First order necessary conditions.

Well posedness

- We define $\Theta : [m, M] \times H^1(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$, where $\theta = \Theta(q, u)$ is the unique solution of the elliptic equation:

$$\nu \nabla^2 \theta - 2q\theta + 2|u|^2 = 0.$$

- Θ is locally Lipschitz continuous.
- Given $q \in \mathcal{Q}$, $u \in C([0, \zeta], H^1(\mathbb{R}^2)$, we have

$$\theta(z) = \Theta(q(z), u(z)) \in C([0, \zeta], H^2(\mathbb{R}^2)).$$

- For the initial value problem with $u(0) = u_0 \in H^1(\mathbb{R}^2)$

$$iu_z + \frac{1}{2}\nabla^2 u + 2\Theta(q, u)u = 0,$$

we prove the existence of a unique solution globally defined.

Well posedness

Theorem

Given $q \in \mathcal{Q}$, the initial value problem for (4) *with* $u(0) = u_0 \in H^1(\mathbb{R}^2)$ *has a unique solution*

$$u[q, u_0] \in C([0, \zeta], H^1(\mathbb{R}^2)) \cap C^1([0, \zeta], H^{-1}(\mathbb{R}^2))$$
$$\theta[q, u_0] \in C([0, \zeta], H^2(\mathbb{R})).$$

Moreover, u is locally Lipschitz continuous.

Regularity of the solution

Theorem

$\Theta(q, u)$ is Fréchet differentiable with respect to q and u.

Theorem

Given $u_0 \in H^1(\mathbb{R}^2)$ and $q \in \mathcal{Q}_{ad}$, let $\theta \in C([0,\zeta], H^2(\mathbb{R}^2))$ and $u \in C([0,\zeta], H^1(\mathbb{R}^2))$ solution of the coupled system. Then u is Fréchet differentiable and $D_q u[q](\delta q) \in C([0,\zeta], H^1(\mathbb{R}^2))$.

Existence of minimum

Theorem

There exists an optimal control $q^* \in \mathcal{Q}_{ad}$.

Sketch of the proof:

Given $q_n \in \mathcal{Q}$ a minimizer's sequence, let $u_n = u[q_n]$ and $\theta_n = \theta[q_n]$ where $\|\dot{q}_n\|_{L^2[0,\zeta]}$ are uniformly bounded. Then, $q_n \rightrightarrows q^*$ in $C[0,\zeta]$, $u_n \to u^*$ in $L^2([0,\zeta], L^2_{loc}(\mathbb{R}^2))$ and $\theta_n \rightharpoonup \theta^*$ in $L^2([0,\zeta], H^2(\mathbb{R}^2))$. Passing to the limit the equations

$$\begin{cases} i(u_n)_z + \frac{1}{2}\nabla^2 u_n + 2\theta_n u_n = 0, \\ \nu\nabla^2\theta_n - 2q_n\theta_n + 2|u_n|^2 = 0, \end{cases}$$

we obtain $q^* \in \mathcal{Q}_{ad}$ is a minimum.

First order necessary conditions

Recall

$$\mathcal{J}(q) = \|u[q](\zeta) - u_1\|^2_{L^2} + \kappa\|\dot{q}\|^2_{L^2(\mathbb{R})}.$$

We can prove that $\mathcal{J}$ is Fréchet differentiable and

$$\begin{aligned}\mathcal{J}'(q)(\delta q) =& 2\langle u[q](\zeta) - u_1, D_q u[q](\zeta)(\delta q)\rangle_{L^2} + 2\kappa\langle \dot{q}, \dot{\delta q}\rangle_{L^2[0,\zeta]} \\ =& 2\langle (D_q u[q](\zeta))^*(u[q](\zeta) - u_1), \delta q\rangle_{L^2[0,\zeta]} + 2\kappa\langle \dot{q}, \dot{\delta q}\rangle_{L^2[0,\zeta]} \\ =& 2\left((D_q u[q](\zeta))^*(u[q](\zeta) - u_1) - \kappa \frac{d}{dt}\dot{q} \right)(\delta q)\end{aligned}$$

Adjoint system

We define

$$\begin{aligned} T : H^1([0,\zeta]) &\longrightarrow \quad H^1(\mathbb{R}^2) \\ \delta q \qquad &\longrightarrow D_q u[q](\delta q)(\zeta) \end{aligned}$$

Then, we characterize the dual operator

$$T^* : H^{-1}(\mathbb{R}^2) \longrightarrow H^{-1}([0,\zeta]))$$

restricted to $H^1(\mathbb{R}^2)$, $T^*(p_\zeta) = -\langle y, \theta \rangle_{L^2(\mathbb{R}^2)} \in L^2[0,\zeta]$ where

$$\begin{aligned} p_z &= i\frac{1}{2}\nabla^2 p(z) + 2\theta p(z) - 2uy \\ p(\zeta) &= p_\zeta \\ -\nu\nabla^2\, y + qy &= -2\mathrm{Re}(\bar{u}p) \end{aligned}$$

First order necessary conditions

Theorem

Let $q^\star \in \mathcal{Q}_{ad}$ be an optimal solution and $u^\star = u[q^\star], \theta^\star = \theta[q^\star]$. Then, $p \in C([0,\zeta], H^1(\mathbb{R}^2)) \cap C^1([0,\zeta], H^{-1}(\mathbb{R}^2))$, $y \in C([0,\zeta], H^2(\mathbb{R}^2))$:

$$
\begin{aligned}
& iu^\star_z = -\frac{1}{2}\nabla^2 u^\star - 2u^\star\theta^\star, \\
& \nu\nabla^2\theta^\star - q^\star\theta^\star = -|u^\star|^2 \\
& u^\star(0) = u_0 \\
& ip_z = -\frac{1}{2}\nabla^2 p(z) + 2ip\theta^\star + 2iu^\star y \\
& \nu\nabla^2\ y - q^\star y = 2\mathrm{Re}(\bar{u}^\star p) \\
& p(\zeta) = u^\star(\zeta) - u_1 \\
& \left(\langle y, \theta^\star\rangle_{L^2} + \frac{d}{dz}\dot{q}^\star\right)(\delta q) \leq 0 \textit{ for all } q \in \mathcal{Q}_{ad}
\end{aligned}
$$

where the last inequality is in the distributional sense.

Future work

- Characterization of an optimal control.
- Numerical approximation.
- Generalization to the more complete models.
- Controllability.

Gracias!