Singular perturbation analysis for a coupled KdV-ODE system

Eduardo Cerpa

Instituto de Ingeniería Matemática y Computacional, UC

January 30, 2023

Joint work with Swann Marx (Nantes)
LACIAM, Rio de Janeiro, January 2023

- Many natural phenomena feature interaction of processes on different times scales.
- A lot of difficulties appear. For instance, huge cost in numerical simulations since the fastest time scale sub-system must be fully solved over a timespan of the slowest scales' order.
- **Desirable**: we want instead solve a limit system, describing approximately the full behavior when some parameters (representing the scales) go to zero (or infinity)

Problem statement

Consider the following coupled system with different **time-scales**

$$
\begin{cases}
\varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\
y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\
y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\
y(0,x) = y_0(x), \ x \in [0,L], \\
\dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\
z(0) = z_0,
\end{cases}
$$

$a, b, c \in \mathbb{R}$, $\varepsilon > 0$ is supposed to be **small**.

Questions

1. What are the **conditions** on a, b, c such that the coupled system is **stable** ? Do these conditions change when ε is small ?
2. What is the **behavior** of the solutions w.r.t. small ε ?

A finite-dimensional example

$$\begin{cases} \varepsilon \dot{y} = ay(t) + bz(t) \\ \dot{z} = cz(t) + dy(t). \end{cases}$$

with $a, b, c, d \in \mathbb{R}$.

Because of ε, the dynamics of y is supposed to be **faster** than the one of $z \Rightarrow$ hence, we fix $a < 0$!

Lyapunov function

$$V(y, z) = \frac{1}{2}\varepsilon y^2 + |\varepsilon M y - z|^2,$$

with $M \in \mathbb{R}$ to be selected. This Lyapunov function is inspired by the **forwarding** approach [Mazenc & Praly, 1996].

$$\dot{V}(y,z) = ay^2 + bzy + (May + Mbz - cz - dy)(\varepsilon My - z)$$

Let us choose M such that $Ma = d$. Hence,

$$\dot{V}(y,z) = ay^2 + bzy + \left(\left(\frac{bd}{a} - c \right) z \right) (\varepsilon My - z)$$

Then, using **Young's inequalities**:

$$\begin{aligned} \dot{V}(y,z) \leq & (a + \alpha_1 + M^2\alpha_2)y^2 \\ & + \left(\frac{\varepsilon^2}{\alpha_2} \left(\frac{bd}{a} - c \right)^2 - \left(\frac{bd}{a} - c \right) + \frac{b^2}{\alpha_1} \right) z(t)^2 \end{aligned}$$

Conditions on a, b, c, d (2)

$$\dot{V}(y,z) = (a + \alpha_1 + M^2\alpha_2)y^2 + \left(\frac{\varepsilon^2}{\alpha_2}\left(\frac{bd}{a} - c\right)^2 - \left(\frac{bd}{a} - c\right) + \frac{b^2}{\alpha_1}\right)z^2$$

Choice of a, b, c, d

1. $a < 0$ and α_1, α_2 sufficiently small so that $a + \alpha_1 + M^2\alpha_2 < 0$
2. b sufficiently small, (a, d, c) satisfying $k_1 < \frac{bd}{a} - c < k_2$, with suitable $k_1, k_2 > 0$ so that the polynomial $\frac{\varepsilon^2}{\alpha_2}X^2 - X + \frac{b^2}{\alpha_1}$ is **always negative**.

Question

If one assumes ε **sufficiently small**, do the conditions change ?

The singular perturbation principle consists in decoupling the coupled system into two approximated systems:

1. The **reduced order system** $\simeq$ **slower system**
2. The **boundary layer system** $\simeq$ **faster system**

Question

How can one compute these two systems ?

Recall that:

$$\begin{cases} \varepsilon \dot{y} = ay(t) + bz(t) \\ \dot{z} = cz(t) + dy(t). \end{cases}$$

Reduced order system

Suppose that $\varepsilon = 0$. Then, $ay + bz = 0 \Rightarrow y = -\frac{b}{a}z$, which is called the **equilibrium point**.

Then, replacing y by the equilibrium point in the z-dynamics, the **reduced order system** reads

$$\dot{\bar{z}} = \left(c - \frac{bd}{a}\right)\bar{z}.$$

Recall that:

$$\begin{cases} \varepsilon \dot{y} = ay(t) + bz(t) \\ \dot{z} = cz(t) + dy(t). \end{cases}$$

Boundary layer system

Set $\tau = \frac{t}{\varepsilon}$ and $\bar{y} = y + \frac{b}{a}z$. Then,

$$\frac{d}{d\tau}\bar{y} = \frac{d}{d\tau}y + \frac{b}{a}\varepsilon\frac{d}{dt}z = a\left(y + \frac{b}{a}z\right) + \frac{b}{a}\varepsilon\frac{d}{dt}z$$

Taking $\varepsilon = 0$, one obtains:

$$\frac{d}{d\tau}\bar{y} = a\bar{y}.$$

Stability conditions

Approximated systems

The reduced order system is

$$\dot{\bar{z}} = \left(c - \frac{bd}{a}\right)\bar{z}.$$

The boundary layer system is

$$\frac{d}{d\tau}\bar{y} = a\bar{y}.$$

Stability conditions

If $a < 0$ and $\left(c - \frac{bd}{a}\right) < 0$, then both systems are stable.

Question

If ε is small enough, do these conditions hold for the **full-system** ?

Change of coordinates

Consider the following change of coordinates:

$$\tilde{y} = y + \frac{b}{a} z,$$

where $-\frac{b}{a} z$ is the **equilibrium point**.
Then, the full-system can be written equivalently as

$$\begin{cases} \varepsilon \dot{\tilde{y}} = a\tilde{y} + \varepsilon \frac{b}{a} \left(\left(c - \frac{bd}{a} \right) z + d\tilde{y} \right), \\ \dot{z} = \left(c - \frac{bd}{a} \right) z + d\tilde{y} \end{cases}$$

Using the Lyapunov function

$$V(\tilde{y}, z) := \frac{1}{2}\varepsilon\tilde{y} + |\varepsilon M\tilde{y} - z|^2$$

one can find ε^* such that, for any $\varepsilon \in (0, \varepsilon^*)$, and for any $a, b, c, d \in \mathbb{R}$ satisfying $a < 0$ and $c - \frac{bd}{a} < 0$, there exist $\mu_1, \mu_2 > 0$ such that

$$\dot{V}(\tilde{y}, z) \leq -\mu_1\tilde{y}^2 - \mu_2 z^2.$$

Generalities

Consider general linear systems:

$$\begin{cases} \varepsilon \dot{y} = Ay + Bz, \\ \dot{z} = Cz + Dy, \end{cases}$$

with $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ and the matrices A, B, C, D of appropriate dimension.

Result

For sufficiently small ε, the conditions for the reduced order system and the boundary layer system to be stable hold for the full-system.

Such a result can be found for instance in [Kokotović, Khalil, O'Reilly, 1986]. The strategy for linear systems relies on a frequency approach. Nonlinear version can be found in [Khalil, 2000].

Question

What about the infinite-dimensional case ?

Existing results and counter-example

Very few results exist for the infinite-dimensional setting:
[Tang, Prieur, Girard, 2015 and 2016],
[Tang, Mazanti, 2017],
[Cerpa, Prieur, 2020].

These results focus on a particular class of systems, namely **hyperbolic systems** coupled with **ODEs**.

Counter-example

$$\begin{cases} \varepsilon \dot{y}(t) = -0.1y(t) - z(1,t) \\ z_t(t,x) + z_x(t,x) = 0 \\ z(0,t) = 2z(1,t) + 0.2y(t). \end{cases}$$

The reduced order system is given by

$$\begin{cases} \bar{z}_t(t,x) + \bar{z}_x(t,x) = 0 \\ \bar{z}(0,t) = 0 \end{cases}$$

and the boundary layer system reads

$$\frac{d}{d\tau}\bar{y}(\tau) = -0.1\bar{y}(\tau).$$

Both systems are **always exponentially stable**. But the full-system is not (proof based on the **method of characteristics**).

Coupled system

Let us go back to the KdV equation coupled with an ODE

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0,\ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0,\ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t),\ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x),\ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0),\ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Here, the fast system is the KdV equation. It should be **exponentially stable without coupling** as in the finite-dimensional case.

Question

What are the stability conditions for a single KdV equation ?

Critical length: exponential stability

$$\begin{cases} y_t + y_x + y_{xxx} = 0,\ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0,\ t \in \mathbb{R}_+, \\ y_x(t,L) = 0,\ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x),\ x \in [0,L]. \end{cases}$$

Theorem (Rosier, 1997)

If $L \notin \mathcal{N}$, with

$$\mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}} : k, l \in \mathbb{N} \right\},$$

The equilibrium point 0 is exponentially stable for the KdV equation.

Moreover, if $L \in \mathcal{N}$, we may loose the **observability property** of the output $y_x(t,0)$.

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = 0, \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L]. \end{cases}$$

Example

The energy $E(y) = \frac{1}{2}\|y\|^2_{L^2(0,L)}$ satisfies

$$\frac{d}{dt}E(y) = -|y_x(t,0)|^2.$$

With $L = 2\pi$ and $y_0(x) = 1 - \cos(x)$, one has

$$y(t,x) = 1 - \cos(x).$$

Thus, $y_x(t,0) = 0$, which implies that $E(y) = E(y_0)$.

Critical length: exponential stability

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = 0, \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L]. \end{cases}$$

For the nonlinear KdV equation:

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = 0, \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L], \end{cases}$$

one can prove the **asymptotic stability** of the origin for some $L \in \mathcal{N}$ ([Tang et al., 2017], [Nguyen, 2020]).

$$\begin{cases} y_t + y_x + y_{xxx} = d_1(t,x), \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = d_2(t), \ t \in \mathbb{R}_+, \end{cases}$$

where d_1 and d_2 are perturbations.

Theorem (Balogoun, Marx, Astolfi, 2022)

Suppose $L \notin \mathcal{N}$. Then, there exists an **ISS Lyapunov functional** $W : L^2(0,L) \to L^2(0,L)$ for the KdV equation, i.e. there exist positive constants $\underline{c}, \overline{c}, \lambda, \kappa_1, \kappa_2, \kappa_3$ such that

$$\underline{c}\|y\|^2_{L^2(0,L)} \leq W(y) \leq \overline{c}\|y\|^2_{L^2(0,L)}$$

and

$$\begin{aligned} \dot{W}(y) \leq & -\lambda\|y\|^2_{L^2(0,L)} + \kappa_1\|d_1(t,\cdot)\|^2_{L^2(0,L)} + \kappa_2|d_2(t)|^2 \\ & -\kappa_3|y_x(t,0|^2 \end{aligned}$$

Strategy

In the following, we will:

- Follow the same procedure as for the finite-dimensional system;
- Use the Lyapunov functional W given by the last theorem.

Main results

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0,\ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0,\ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t),\ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x),\ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0),\ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Proposition (Marx and C., 2023)

For any $\varepsilon > 0$, there exist a_*, k_1, k_2 such that if $a < a_*$ and b, c satisfy $k_1 < ac - b < k_2$, then the origin is globally exponentially stable.

This result can be seen as a sort of generalization of one of the result in [Balogoun, Marx, Astolfi, 2022], where $b = 0$ and $c = 1$.

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Theorem (Marx and C., 2023)

For any $a, b, c \in \mathbb{R}$ such that $(b - ac) < 0$, there exists ε^* such that, for any $\varepsilon \in (0, \varepsilon^*)$, the origin is globally exponentially stable.

We will see that the singular perturbation method applies for the coupled KdV-ODE system !

Suppose that $\varepsilon = 0$. Then,

$$\begin{cases} h_x + h_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ h(t,0) = h(t,L) = 0, \ t \in \mathbb{R}_+, \\ h_x(t,L) = az(t), \ t \in \mathbb{R}_+. \end{cases}$$

There exists an explicit solution to this problem:

$$h(t,x) = -2az(t)f(x),$$

with $f(x) = \frac{1}{\sin\left(\frac{L}{2}\right)} \sin\left(\frac{x}{2}\right) \sin\left(\frac{L-x}{2}\right)$. Note that $h_x(t,0) = -az(t)$, then replacing in $\dot{z} = bz(t,0) + cy_x(t,0)$ $y_x(t,0)$ by $-az$, one obtains

Reduced order system

$$\dot{\bar{z}}(t) = (b - ac)\bar{z}(t).$$

If $(b - ac) < 0$, then this system is **exponentially stable** !

Set $\tau = \frac{t}{\varepsilon}$. After some computations similar to the finite-dimensional case, one obtains:

Boundary layer system

$$\begin{cases} y_\tau + y_x + y_{xxx} = 0, \\ y(\tau, 0) = y(\tau, L) = 0, \\ y_x(\tau, L) = 0. \end{cases}$$

If $L \notin \mathcal{N}$, the system is **exponentially stable** !

As for the finite-dimensional case, we will follow a **forwarding approach**, i.e. we consider

$$V(y,z) = \varepsilon W(y) + \frac{1}{2}(\varepsilon \mathcal{M} y - z)^2,$$

which is the same Lyapunov functional as in [Balogoun, Marx, Astolfi, 2020]. The operator $\mathcal{M}$ is an **integral operator**, i.e.

$$\mathcal{M} y = \int_0^L M(x) y(x) dx,$$

where M is the solution to

$$\begin{cases} M'''(x) + M'(x) = 0, \\ M(0) = M(L) = 0, \\ M'(0) = -c, \end{cases}$$

with $M(x) = -f(x)c$.

Recall that:

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0,\ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0,\ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t),\ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x),\ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0),\ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Using the ISS Lyapunov functional

Seeing $az(t)$ as a perturbation, one has

$$\varepsilon \dot{W}(y) \leq -\lambda \|y\|^2_{L^2(0,L)} + \kappa_2 a^2 z(t)^2$$

Time-derivative of the Lyapunov functional

Recall that:

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Differentiating the other term

Integration by parts + M + Young's inequality

$$\frac{d}{dt}\frac{1}{2}\left(\varepsilon \int_0^L M(x)y(t,x)dx - z(t)\right)^2 \leq$$

$$(b-ac)z(t)^2 + \alpha\varepsilon^2 \|M\|^2_{L^2(0,L)} \|y\|^2_{L^2(0,L)} + \frac{(b-ac)^2}{\alpha} z(t)^2$$

One finally has:

$$\begin{aligned}\dot{V}(y,z) \leq & (-\lambda + \alpha\varepsilon^2 \|M\|^2_{L^2(0,L)})\|y\|^2_{L^2(0,L)} \\ & + \left(\frac{(b-ac)^2}{\alpha} + (b-ac) + \kappa_2 a^2\right) z(t)^2\end{aligned}$$

Choose

- α so that $-\lambda + \alpha\varepsilon^2 \|M\|^2_{L^2(0,L)} < 0$
- a sufficiently small and $k_1 < (ac-b) < k_2$, $k_1, k_2 > 0$, so that the polynomial $\frac{X^2}{\alpha} - X + \kappa_2 a^2$ is always negative.

Small ε: change of coordinates

We consider

$$\tilde{y}(t,x) = y(t,x) + 2f(x)az(t)$$

One obtains

$$\begin{cases} \varepsilon\tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = -\varepsilon((b-ac)z(t) + c\tilde{y}_x(t,0))f(x) \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0 \\ \tilde{y}_x(t,L) = 0 \\ \dot{z} = (b-ac)z + c\tilde{y}_x(t,0). \end{cases}$$

Using the same Lyapunov functional as before, one obtains the desired result !

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \end{cases}$$

Theorem (Marx and C., 2023)

There exist $a_*, k_1, k_2, \varepsilon^*, \mu > 0$ such that if $a < a_*$, b, c satisfy $0 < k_1 < -(b - ac) < k_2$ and $\varepsilon < \varepsilon^*$, then with any initial condition satisfying

$$\|y_0 - \bar{y}_0 + f z_0\|_{L^2(0,L)} + |z_0 - \bar{z}_0| = O(\varepsilon^{\frac{3}{2}}), \qquad |\bar{z}_0| = O(\varepsilon^{\frac{1}{2}})$$
$$\|\bar{y}_0\|_{L^2(0,L)} = O(\varepsilon^{\frac{3}{2}}),$$

one has

$$\|y(t,\cdot) - \bar{y}(t/\varepsilon,\cdot) + f(\cdot)z(t)\|_{L^2(0,L)} + |z(t) - \bar{z}(t)| = O(\varepsilon)e^{-\mu t}.$$

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ \varepsilon \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \end{cases}$$

Question

Does the singular perturbation method apply for the case where the ODE is fast ?

Answer: yes, but some adjustments need to be done !

Reduced order system

Set $\varepsilon = 0$, one has $z(t) = -\frac{c}{b} y_x(t, 0)$.

$$\begin{cases} \bar{y}_t + \bar{y}_x + \bar{y}_{xxx} = 0 \\ \bar{y}(t, 0) = \bar{y}(t, L) = 0 \\ \bar{y}_x(t, L) = -\dfrac{ac}{b} \bar{y}_x(t, 0). \end{cases}$$

Boundary layer system

$$\frac{d}{d\tau} \bar{z}(\tau) = b\bar{z}(\tau),$$

with $\tau = \frac{t}{\varepsilon}$ and $\bar{z} = z + \frac{c}{b} y_x(t, 0)$.

Stability conditions: $b < 0$ (obvious) and $\left|\frac{ac}{b}\right| < 1$ (not so obvious, but known [Zhang, 1994]).

Recall that:

$$\dot{W}(y) \leq -\lambda\|y\|^2_{L^2(0,L)} + \kappa_1\|d_1(t,\cdot)\|^2_{L^2(0,L)} + \kappa_2|d_2(t)|^2 - \kappa_3|y_x(t,0|^2$$

Theorem (Marx and C., 2023)

$\forall a, b, c \in \mathbb{R}$ such that $\frac{a^2c^2}{b^2} < \frac{\kappa_3}{\kappa_2}$, where κ_2 and κ_3 are defined in the definition of W, $\exists \varepsilon^* > 0$ such that $\forall \varepsilon \in (0, \varepsilon^*)$, the origin is exponentially stable with initial conditions $(y_0, z_0) \in H^3(0,L) \times \mathbb{R}$ such that

$$y_0(0) = y_0(L) = 0, \quad y_0'(L) = ab(z_0 + \frac{c}{b}y_0'(0)).$$

Question

Why the initial conditions need to be **so regular** ? As we use $\tilde{z} = z + \frac{c}{b}y_x(t,0)$, we differentiate $y_x(t,0)$ with respect to time, which requires higher regularity !

Achievements and open problems

Achievements

1. We have applied the **singular perturbation analysis** for a coupled KdV-ODE system;
2. Special Lyapunov functionals have been designed to achieve such a result.

Open problems

1. What about the case of **coupled PDEs** ? Work in progress on the parabolic-hyperbolic case with Gonzalo Arias (PhD student, UC) and Swann Marx.
2. What about the case of **operators generating semigroups** ? Is it possible to find a general result ?
3. Other counterexamples where the approach fails.

Thank you for your attention