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STABILIZATION RESULTS FOR DELAYED FIFTH-ORDER KDV-TYPE EQUATION IN A BOUNDED DOMAIN

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Outline

1 Introduction

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This talk is doveted to the analysis of solutions of the Kawahara equation [18, 24], a fifth higher-order Korteweg-de Vires (KdV) equation

$$u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = 0 \tag{1}$$

which is a dispersive PDE describing numerous wave phenomena such as magneto-acoustic waves in a cold plasma [22], gravity waves on the surface of a heavy liquid [13], etc.

The objective here is to analyze the qualitative properties of solutions to the initial-boundary value problem for (1) posed on a bounded interval under the presence of a localized damping and delay terms, that is

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) - u_{xxxxx}(x,t) & \\ +u(x,t)u_x(x,t) + a\,(x)\,u(x,t) + b(x)u(x,t-h) = 0 & x \in (0,L),\ t > 0, \\ u\,(0,t) = u\,(L,t) = 0 & t > 0, \\ u_x\,(0,t) = u_x\,(L,t) = u_{xx}\,(L,t) = 0 & t > 0, \\ u\,(x,0) = u_0\,(x) & x \in (0,L), \\ u(x,t) = z_0(x,t) & x \in (0,L),\ t \in (-h,0), \end{cases} \tag{2}$$

where $h > 0$ is the time delay, $L > 0$ is the length of the spatial domain, $u(x,t)$ is the amplitude of the water wave at position x at time t, and $a = a(x)$ and $b = b(x)$ are nonnegative functions belonging to $L^\infty(\Omega)$.

For our purpose let us introduce the following assumption.

Assumption 1

The real functions $a = a(x)$, $b = b(x)$ *are nonnegative functions belonging to* $L^{\infty}(\Omega)$. *Moreover,* $a(x) \geq a_0 > 0$ *almost everywhere in a nonempty open subset* $\omega \subset (0, L)$.

Note that the term $a(x)u$ designs a feedback damping mechanism (see, for instance [1]); therefore, one can expect the global well-posedness of (2) for all $L > 0$. Therefore, defining the energy of system (2) by

$$E_u(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{h}{2}\int_0^L\int_0^1 b(x)u^2(x,t-\rho h)d\rho dx, \tag{3}$$

the following questions arise:

Does $E_u(t) \longrightarrow 0$, as $t \to \infty$? If it is the case, can we give a decay rate?

In this case, the derivative of the energy E satisfies

$$\begin{aligned}\frac{d}{dt}E_u(t) \leq & -u_{xx}^2(0) - \int_0^L a(x)u^2(x,t)dx + \frac{1}{2}\int_0^L b(x)u^2(x,t)dx \\ & + \frac{1}{2}\int_0^L b(x)u^2(x,t-h)dx + \frac{1}{2}\int_0^L b(x)u^2(x,t)dx \\ & - \frac{1}{2}\int_0^L b(x)u^2(x,t-h)dx \\ \leq & \int_0^L b(x)u^2(x,t)dx.\end{aligned}$$

The previous inequality means that the energy is not decreasing in general, since the term $b(x) \geq 0$ on $(0, L)$.

Theorem 1

Assume that the functions $a(\cdot)$ *and* $b(\cdot)$ *satisfy the conditions given in Assumption 1 and let* $L < \pi\sqrt{3}$. *Under these assumptions, there exist* $\delta > 0$, $r > 0$, $C > 0$ *and* $\nu > 0$, *such that if* $\|b\|_\infty < \delta$, *then for every* $(u_0, z_0) \in \mathcal{H} = L^2(0, L) \times L^2((0, L) \times (0, 1))$ *satisfying* $\|(u_0, z_0)\|_{\mathcal{H}} \leq r$, *the energy* (3) *of the system* (2) *satisfies*

$$E_u(t) \leq Ce^{-\nu t} E_u(0), \text{ for all } t \geq 0.$$

Another goal of this talk, is to consider the following μ_i-system

$$\begin{cases} u_t(x,t)+u_x(x,t)+u_{xxx}(x,t)-u_{xxxxx}(x,t) & \\ +u(x,t)u_x(x,t)+a\left(x\right)\left(\mu_1 u(x,t)+\mu_2 u(x,t-h)\right)=0 & x\in(0,L),\ t>0, \\ u\left(0,t\right)=u\left(L,t\right)=0 & t>0, \\ u_x\left(0,t\right)=u_x\left(L,t\right)=u_{xx}\left(L,t\right)=0 & t>0, \\ u\left(x,0\right)=u_0\left(x\right) & x\in(0,L), \\ u(x,t)=z_0(x,t) & x\in(0,L),\ t\in(-h,0), \end{cases} \tag{4}$$

which was inspired by the work of Nicaise and Pignotti [31].

Here $h > 0$ is the time delay, $\mu_1 > \mu_2$ are positive real numbers. We consider the following energy associated with the solutions of the system (4)

$$E_u(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{\xi}{2}\int_0^L\int_0^1 a(x)u^2(x,t-\rho h)d\rho dx, \tag{5}$$

where ξ is a positive constant verifying the following

$$h\mu_2 < \xi < h(2\mu_1 - \mu_2). \tag{6}$$

Again, we are interested to see the questions previously mentioned. Note that, in a different way from our first goal, the derivative of the energy (5) satisfies

$$E_u'(t) \leq -C\left[u_{xx}^2(0) + \int_0^L a(x)u^2(x)dx + \int_0^L a(x)u^2(x,t-h)dx\right] \leq 0,$$

for some positive constant $C := C(\mu_1, \mu_2, \xi, h)$.

For the system (4) we split the behavior of the solutions into two parts. Employing Lyapunov's method, it can be deduced that the energy (5) goes exponentially to zero as $t \to \infty$, however, the initial data needs to be sufficiently small in this case. Precisely, the local result can be read as follows.

Theorem 2

Let $L > 0$*, assume that* $a \in L^\infty(\Omega)$*,* (6) *holds and* $L < \pi\sqrt{3}$*. Then, there exists* $0 < r < \frac{9\pi^2 - 3L^2}{2L^{\frac{3}{2}}\pi^2}$ *such that for every* $(u_0, z_0(\cdot, -h(\cdot))) \in \mathcal{H}$ *satisfying* $\|(u_0, z_0(\cdot, -h(\cdot)))\|_{\mathcal{H}} \leq r$*, the energy* (5) *of the system* (4) *decays exponentially. More precisely, considering*

$$\gamma = \min\left\{\frac{9\pi^2 - 3L^2 - 2L^{\frac{3}{2}}r\pi^2}{3L^2(1+2L\alpha)}\alpha, \frac{\beta\xi}{2h(\xi\beta+\xi)}\right\} \quad \text{and} \quad \kappa = (1+\max\{2\alpha L, \beta\}),$$

with α *and* β *positive constants such that*

$$\alpha < \min\left\{\frac{1}{2L\mu_1 + L\mu_2}\left(\mu_1 - \frac{\xi}{2h} - \frac{\mu_2}{2} - \frac{\beta\xi}{2h}\right), \frac{1}{L\mu_2}\left(\frac{\xi}{2h} - \frac{\mu_2}{2}\right)\right\},$$
$$\beta < \frac{2h}{\xi}\left(\mu_1 - \frac{\xi}{2h} - \frac{\mu_2}{2}\right).$$

Then,

$$E(t) \leq \kappa E(0)e^{-2\gamma t} \text{ for all } t > 0.$$

The last result of the talk, still related to the system (4), we give the result that removes the hypothesis of the initial data being small. To do that, we use the compactness-uniqueness argument, which reduces our problem to prove an *observability inequality* for the nonlinear system (4). More precisely, we have the following semi-global result.

Theorem 3

Assume that $a(x)$ satisfies Assumption 1. Suppose that $\mu_1 > \mu_2$ and let $\xi > 0$ satisfying (6). Let $R > 0$, then there exists $C = C(R) > 0$ and $\nu = \nu(R) > 0$ such that E_u, defined in (5), satisfies

$$E_u(t) \leq C E_u(0) e^{-\nu t}, \quad \forall t > 0,$$

for solutions of (4) provided that $\|(u_0, z_0)\|_{\mathcal{H}} \leq R$.

1 Introduction
- Previous results

2 Study of the damping–delayed system

Concerning the Kawahara equation, recently in [1], the authors considered the following damped system

$$u_t + u_x + u_{xxx} - u_{xxxxx} + u^p u_x + a(x)u = 0, \quad (x,t) \in (0,L) \times (0,T), \tag{7}$$

for $p \in [1,4)$, with a presence of an extra damping term $a(x)$ satisfying Assumption 1.

This damping mechanism is essential already in a linear case: if $a(x) \equiv 0$ and the length of an interval is critical (see [1]), then it can be constructed a nontrivial solution to

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = 0, & (x,t) \in (0,L) \times (0,T) \\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & t \in (0,T) \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases}$$

which does not decay to 0 as $t \to \infty$.

Inspired by [41, J. Valein], we consider the following perturbed system

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) - u_{xxxxx}(x,t) & \\ +u(x,t)u_x(x,t) + a\,(x)\,u(x,t) & \\ +b(x)u(x,t-h) + b(x)\xi u(x,t) = 0 & x \in (0,L),\ t > 0, \\ u\,(0,t) = u\,(L,t) = 0 & t > 0, \\ u_x\,(0,t) = u_x\,(L,t) = u_{xx}\,(L,t) = 0 & t > 0, \\ u\,(x,0) = u_0\,(x) & x \in (0,L), \\ u(x,t) = z_0(x,t) & x \in (0,L),\ t \in (-h,0), \end{cases} \tag{8}$$

which is “close” to (2) but with a nonincreasing energy, with ξ a positive constant.

First, let us consider the system (8) linearized around 0.

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) - u_{xxxxx}(x,t) & \\ +a\,(x)\,u(x,t) + b(x)u(x,t-h) + \xi b(x)u(x,t) = 0 & x \in (0,L),\ t>0, \\ u\,(0,t) = u\,(L,t) = 0 & t>0, \\ u_x\,(0,t) = u_x\,(L,t) = u_{xx}\,(L,t) = 0 & t>0, \\ u\,(x,0) = u_0\,(x) & x \in (0,L), \\ u(x,t) = z_0(x,t) & x \in (0,L),\ t \in (-h,0), \end{cases} \tag{9}$$

with the energy defined by

$$E_u(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{\xi h}{2}\int_0^L\int_0^1 b(x)u^2(x,t-\rho h)\rho dx, \tag{10}$$

For the perturbed system we get, for $\xi > 1$, that the derivative of the energy $E_u(t)$, for classical solutions of (8), satisfies

$$
\begin{aligned}
\frac{d}{dt}E_u(t) \leq & -u_{xx}^2(0) - \int_0^L a(x)u^2(x,t)dx + \frac{1}{2}\int_0^L b(x)u^2(x,t)dx \\
& + \frac{1}{2}\int_0^L b(x)u^2(x,t-h)dx - \int_0^L \xi b(x)u^2(x,t)dx \\
& + \frac{1}{2}\int_0^L \xi b(x)u^2(x,t)dx - \frac{1}{2}\int_0^L \xi b(x)u^2(x,t-h)dx \\
\leq & -u_{xx}^2(0) - \int_0^L a(x)u^2(x,t)dx + \frac{1}{2}\int_0^L (b(x)-\xi b(x))u^2(x,t)dx \\
& + \frac{1}{2}\int_0^L (b(x)-\xi b(x))u^2(x,t-h)dx \leq 0.
\end{aligned}
$$

Choosing the following Lyapunov functional

$$V(t) = E(t) + \alpha V_1(t) + \beta V_2(t), \tag{11}$$

where α and β are positive constants that will be fixed small enough, later on, where V_1 is defined by

$$V_1(t) = \int_0^L x u^2(x,t)dx \tag{12}$$

and V_2 is defined by

$$V_2(t) = \frac{h}{2}\int_0^L \int_0^1 (1-\rho)b(x)u^2(x,t-\rho h)d\rho dx. \tag{13}$$

Proposition 2.1

Assume that a and b are nonnegative function in $L^\infty(0,L)$, $b(x) \geq b_0 > 0$ in ω, $L < \pi\sqrt{3}$ and $\xi > 1$. Then, for every $(u_0, z_0(\cdot, -h(\cdot))) \in \mathcal{H}$, the energy of system (9), denoted by E and defined by (10), decays exponentially. More precisely, considering

$$\gamma = \min\left\{\frac{(3\pi^2 - L^2)\alpha}{L^2(1+2\alpha L)}, \frac{\beta}{2h(\xi+\beta)}\right\} \text{ and } \kappa = \left(1 + \max\left\{2\alpha L, \frac{\beta}{\xi}\right\}\right),$$

where α is a positive constant such that

$$\alpha < \frac{\xi - 1}{2L(1+2\xi)}$$

and

$$\beta = \xi - 1 - 2\alpha L(1+2\xi),$$

then

$$E(t) \leq \kappa E(0) e^{-2\gamma t} \text{ for all } t > 0.$$

Now we will study the asymptotic stability of the linear system associated with (8), namely,

$$\begin{cases} u_t(x,t)+u_x(x,t)+u_{xxx}(x,t)-u_{xxxxx}(x,t) & \\ +b(x)u(x,t-h)+a(x)\,u(x,t)=0 & x\in(0,L),\ t>0, \\ u(0,t)=u(L,t)=0 & t>0, \\ u_x(0,t)=u_x(L,t)=u_{xx}(L,t)=0 & t>0, \\ u(x,0)=u_0(x) & x\in(0,L), \\ u(x,t)=z_0(x,t) & x\in(0,L),\ t\in(-h,0). \end{cases} \tag{14}$$

The next result ensures that the energy

$$E_u(t)=\frac{1}{2}\int_0^L u^2(x,t)dx+\frac{h}{2}\int_0^L\int_0^1 b(x)u^2(x,t-\rho h)d\rho dx, \tag{15}$$

associated of the system (14) decays exponentially

Proposition 2.2

Assume that a *and* b *are nonnegative function in* $L^\infty(0,L)$, $b(x) \geq b_0 > 0$ *in* ω, $L < \pi\sqrt{3}$ *and* $\xi > 1$. *So, there exists* $\delta > 0$ *(depending on* ξ, L, h*) such that if,* $\|b\|_\infty \leq \delta$ *then, for every* $(u_0, z_0(\cdot, -h(\cdot))) \in \mathcal{H}$ *the energy of system* E_u, *defined in* (15), *goes to* 0 *exponentially as* t *goes to infinity. More precisely, there exists* $T_0 > 0$ *and two positive constants* ν *and* C *such that*

$$E_u(t) \leq Ce^{-\nu t} E_u(0), \ \textit{for all } t \geq 0.$$

To prove this result, let us consider the two systems

$$\begin{cases} v_t(x,t) + v_x(x,t) + v_{xxx}(x,t) - v_{xxxxx}(x,t) & \\ +a\,(x)\,v(x,t) + b(x)z^1(1) + \xi b(x)v(x,t) = 0 & x \in (0,L),\ \ t > 0, \\ v\,(0,t) = v\,(L,t) = 0 & t > 0, \\ v_x\,(0,t) = v_x\,(L,t) = v_{xx}\,(L,t) = 0 & t > 0, \\ v\,(x,0) = u_0\,(x) & x \in (0,L), \\ hz^1_t(x,\rho,t) + z^1_\rho(x,\rho,t) = 0 & x \in (0,L),\ \ \rho \in (0,1),\ \ t > 0, \\ z^1(x,0,t) = v(x,t) & x \in (0,L),\ \ t > 0, \\ z^1(x,\rho,0) = v(x,-\rho h) = z_0(x,-\rho h) & x \in (0,L),\ \ \rho \in (0,1) \end{cases} \tag{16}$$

and

$$\begin{cases} w_t(x,t) + w_x(x,t) + w_{xxx}(x,t) - w_{xxxxx}(x,t) & \\ +a\,(x)\,w(x,t) + b(x)z^2(1) = \xi b(x)v(x,t) & x \in (0,L),\ t>0, \\ w\,(0,t) = w\,(L,t) = 0 & t>0, \\ w_x\,(0,t) = w_x\,(L,t) = w_{xx}\,(L,t) = 0 & t>0, \\ w\,(x,0) = 0 & x \in (0,L), \\ hz^2_t(x,\rho,t) + z^2_\rho(x,\rho,t) = 0 & x \in (0,L),\ \rho \in (0,1),\ t>0, \\ z^2(x,0,t) = w(x,t) & x \in (0,L),\ t>0, \\ z^2(x,\rho,0) = 0 & x \in (0,L),\ \rho \in (0,1). \end{cases} \tag{17}$$

Define $u = v + w$ and $z = z^1 + z^2$, then

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) - u_{xxxxx}(x,t) & \\ +a\,(x)\,u(x,t) + b(x)z(1) = 0 & x \in (0,L),\ t > 0, \\ u\,(0,t) = u\,(L,t) = 0 & t > 0, \\ u_x\,(0,t) = u_x\,(L,t) = u_{xx}\,(L,t) = 0 & t > 0, \\ u\,(x,0) = u_0\,(x) & x \in (0,L), \\ hz_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & x \in (0,L),\ \rho \in (0,1),\ t > 0, \\ z(x,0,t) = u(x,t) & x \in (0,L),\ t > 0, \\ z(x,\rho,0) = z_0(x,-\rho h) & x \in (0,L),\ \rho \in (0,1). \end{cases} \tag{18}$$

Fix $0 < \eta < 1$ and pick

$$T_0 = \frac{1}{2\gamma} \ln\left(\frac{2\xi\kappa}{\eta}\right) + 1, \tag{19}$$

so $\kappa e^{-2\gamma T_0} < \frac{\eta}{2\xi}$ and $E_v(T_0) \leq \kappa E_v(0) e^{-2\gamma T_0} < \frac{\eta}{2\xi} E_v(0) < \frac{\eta}{2} E_u(0)$.

After some computations, we have

$$\begin{aligned} E_u(T_0) \leq & 2E_v(T_0) + 2\xi^3 \|b\|_\infty^2 e^{(3\xi+1)T_0} \kappa E_u(0) \\ \leq & (\eta + \varepsilon) E_u(0), \end{aligned}$$

where $\varepsilon > 0$ is such that $\eta + \varepsilon < 1$. Proceeding in an analogous way we get

$$E_u(mT_0) \leq (\eta + \varepsilon)^m E_u(0),$$

for all $m \in \mathbb{N}^*$.

Now, to finish, let $t > T_0$, then there exists $m \in \mathbb{N}^*$ such that $t = mT_0 + s$ with $0 \leq s < T_0$, we have

$$\begin{aligned}
E_u(t) \leq & e^{2\|b\|_\infty (t-mT_0)} E_u(mT_0) \\
\leq & e^{2\|b\|_\infty s} (\eta + \varepsilon)^m E_u(0) \\
= & e^{2\|b\|_\infty s} e^{-\nu m T_0} E_u(0) \\
= & e^{2\|b\|_\infty s} e^{-\nu (t-s)} E_u(0) \\
\leq & e^{(2\|b\|_\infty + \nu) T_0} e^{-\nu t} E_u(0),
\end{aligned}$$

where

$$\nu = \frac{1}{T_0} \ln \left(\frac{1}{(\eta + \varepsilon)} \right), \tag{20}$$

showing the proposition.

Lemma 4

For any $T, R > 0$ *there exists* $K := K(R,T) > 0$ *such that*

$$\|u\|^2_{L^2(0,T,L^2(0,L))} \leq K\left(\int_0^T u_{xx}^2(0,t)dt + \int_0^T\int_0^L a(x)u^2(x)dxdt + \int_0^T\int_0^L a(x)u^2(x,t-h)dxdt\right) \tag{21}$$

holds for all solutions of the nonlinear system (4) *with* $\|(u_0, z_0(\cdot,-h(\cdot)))\|_{\mathcal{H}} \leq R$.

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