ZERO BLACK DERMAN TOY MODEL

Andrés Sosa

Instituto de Estadística – DMC – FCEA
Universidad de la República

Latin American Congress on Industrial and Applied Mathematics
Rio de Janiero - Brazil

This is joint work with Ernesto Mordecki and Grzegorz Krzyżanowski

andres.sosa@fcea.edu.uy

February 2, 2023

Objective, Motivation and Contents

Objective

Introduce a ***new interest rate model*** *called* ***Zero Black Derman Toy Model****.*

Motivation

This proposal is motivated by the ***sharp falls in interest rates*** *in the United States during the 2008 financial crisis.*

Contents

1. *Federal Funds Rate and ZIRP policy*
2. *Black Derman Toy Model*
3. *Zero Black Derman Toy Model*
4. *Two applications in the United State recession*
    1. *2008 recession*
    2. *Covid-19 recession*

Federal Fund Rate

- The **Federal Funds Rate** is the benchmark interest rate that applies to loans among commercial banks over a period of time (overnight). This benchmark (or range) is set by the Federal Open Market Committee (FOMC), which is the policymaking authority of the Federal Reserve.

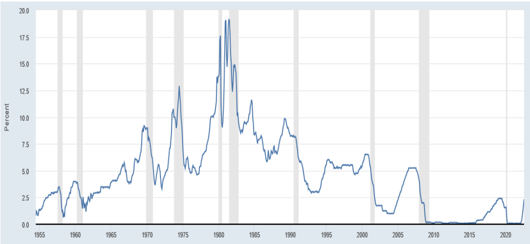
- This interest rate is intended to affect the country's financial conditions, influencing important aspects of the economy such as employment, inflation and interest rates, among others.

Zero Interest Rate Policy

In response to the 2008 crisis, the United States ***reduced the interest rate*** *by 425 basis points in less than a year.*

This policy was also applied at the beginning of the pandemic generated by Covid-19.

Historical Federal Funds Effective Rate (1955-2022)



In gray, years of U.S. recession

Different Approaches to Model the ZIRP

After the 2008 crisis, several approaches to modeling ZIRP emerged.

2016 **Lewis:** Propose a slow reflection limit. This model consists on the introduction of *sticky points* that retain the process longer than the other points.

2016 **Lewis:** The model proposes the introduction of a time delay by means of an exponential process.

2018 **Tian and Zhang:** Based in the classic CIR process, the model adds one skew point at a certain relatively small level of the interest rate.

2018 **Eberlein et al.:** This proposal is in the context of Lévy modeling of Libor rates, and the modification allows negative interest rates. The model is presented in the framework of the semimartingale theory.

2018 **Martin:** The proposal is to use several interest rate curves in the same model, which reflect the different types of risk observed in the fixed income markets. The model is based in the intensity models.

Our Proposal

Proposal

- *Our approach consists mainly in* ***modeling for the purpose of assets pricing*** *rather than in view of hedging and/or risk management.*

- *Therefore, the idea is to* ***modify the BDT model (binary tree model) to a mixed binary/ternary tree model*** *to find consistent interest rates with the market using the same data.*

In the modifications we use tools similar to:

- *Modeling Term Structure of Defaultable Bonds.*
  Duffie, D., Singleton, K. (1999),

- *Option Valuation under Stochastic Volatility II.*
  Lewis, A. (2016),

Why did we choose to modify the BDT model?

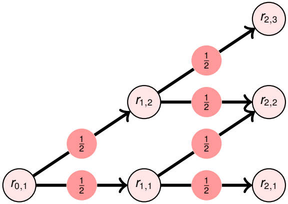
We selected this model because it has good properties:

- It is one of the most popular fixed income valuation models.
- It was developed for internal use by *Goldman Sachs* in the 1980s.
- It was published in the *Financial Analysts Journal* in 1990.
- This model is part of the typology known as tree models.
- Combines mean-reverting behaviour and lognormal distribution.
- The bond prices in the model coincide with market prices.
- The data required for calibration are only historical and current yield curves.
- Presents efficient algorithms for the estimation of interest rates.
- **It is easy to carry out asset pricing.**

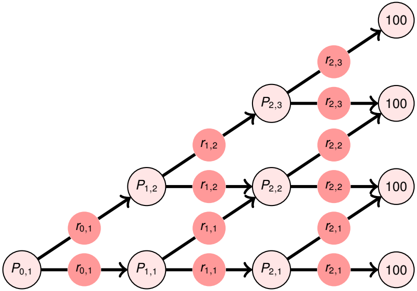
Black Derman Toy Model (BDT Model)

- The model assumes that the future interest rates evolve randomly in a binomial tree at **each node of probability** $1/2$.
- After the n periods, the model has $n+1$ possible states for our stochastic process modeling the interest rate.

INTEREST RATE TREE



PRICE BOND TREE



Calibration of the BDT Model

Required Market Information:

- The **current market yield curve**, i.e. the yield rates for n periods $y(k),\ k = 1, \dots, n$
- The **historical market yield curve** to estimate their volatilities $\beta(k),\ k = 2, \dots, n$.

Calibration Assumptions:

- For the bond with maturity at time k, the market price **is equal to** the model price.
- **Volatility** depends only on time, not on the state of interest rates.

  That is, there is a single volatility at the same time for all possible interest rates.

Calibration Procedure:

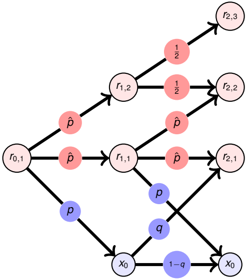
- The **forward induction procedure** is performed, which consists of accumulating the interest rates as we move through the tree.

  That is, the value for all states in time k is obtained from all estimated values of the interest rate up to time $k-1$ and the model assumptions.

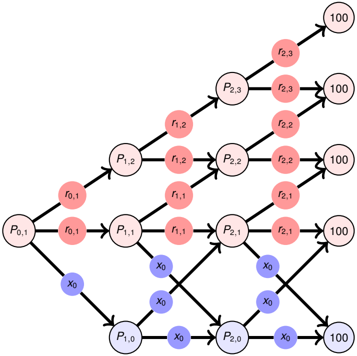
Zero Black Derman Toy Model (ZBDT Model)

Our modification of the BDT model adds **a small probability of a downward jump** to a practically zero interest rate, which if reached the process remains with high probability.

Interest Rate Tree



Price Bond Tree



The model adds three parameters:

- p: it refers to probability of the crisis in the next time,
- q: it refers to conditional probability of economic recovery from the financial crisis,
- x_0: it refers to the value of the interest rate in the ZIRP zone.

Zero Black Derman Toy Model (ZBDT Model)

In more detail, the properties of the knots are:

- knots (i, j) with $j \geq 2$ have the same characteristics than the BDT model (probability $1/2$ for each new knots),
- knots $(1, j)$ add a jump down with small probability p and the other two have probability $\hat{p} = (1 - p)/2$,
- in this jump the interest rate becomes a default value x_0 (near the ZIRP target),
- when the process is in this state, it has a (high) probability of remaining $(1 - q)$ and a (low) probability of exiting q.

The aspects on the calibration are similar to the previous ones:

- **Required Market Information:** the same as in BDT model.
- **Calibration Assumptions:** similar to BDT model.
- **Calibration Procedure:** the system is **more complex** to solve.

Two applications in the United State recessions

Objective

*We analyse **the performance** of the ZBDT model and compare the results with the BDT model in different contexts of the United States economy*

1. **2008 recession**
   - We analyse different economic situations between 2002 and 2016.
   - We show the results before the crisis (August 2006).
   - We show the results after the crisis (May 2015).
   - We only consider the Vanilla Options.
2. **Covid-19 recession**
   - We analyse every day of 2020.
   - We show the results before the crisis (February 2020).
   - We consider a large number of exotic financial derivatives.
   - We consider European and American style.

Applications are carried out for five years

Payoff functions for the selected derivatives (European style)

Derivative	Payoff Functions
Vanilla Option	$\max(Z_S - K, 0)$
Knock up-and-in	$\max(Z_S - K, 0)\,\mathbb{1}\{M_S > H^+\}$
Knock up-and-out	$\max(Z_S - K, 0)\,\mathbb{1}\{M_S \leq H^+\}$
Knock down-and-in	$\max(Z_S - K, 0)\,\mathbb{1}\{m_S < H^-\}$
Knock down-and-out	$\max(Z_S - K, 0)\,\mathbb{1}\{m_S \geq H^-\}$
Knock double-out	$\max(Z_S - K, 0)\,\mathbb{1}\{M_S \leq H^+, m_S \geq H^-\}$
Knock double-in	$\max(Z_S - K, 0) - \max(Z_S - K, 0)\,\mathbb{1}\{M_S \leq H^+, m_S \geq H^-\}$

where

- T is the maturity of the bond,
- S is the expiry time of the option,
- K is strike,
- $M_t = \max_{\tau \in [0,t]} (Z_\tau)$,
- $m_t = \min_{\tau \in [0,t]} (Z_\tau)$
- H^+, H^- are upper and lower barriers respectively.

First Case: 2008 Recession

Before the crisis: August 7, 2006

				8.41
			7.12	6.40
		6.37	5.51	4.88
	5.66	4.77	4.26	3.71
4.97	3.96	3.57	3.30	2.83

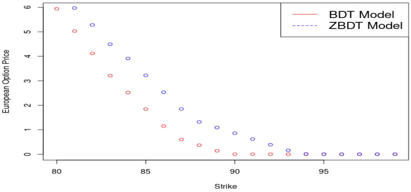
				17.79
			12.23	8.58
		9.06	6.08	4.14
	6.70	4.32	3.02	2.00
4.97	3.00	2.13	1.54	0.98
	0.25	0.25	0.25	0.25

After the crisis: May 20, 2015

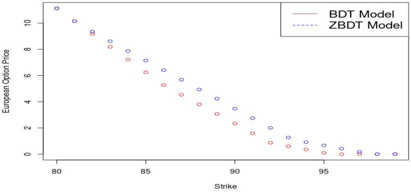
				7.63
			6.52	4.78
		5.34	3.72	2.99
	2.97	2.35	2.13	1.87
0.61	0.92	1.04	1.21	1.17

				16.66
			11.02	6.01
		6.91	3.70	2.17
	3.21	1.91	1.24	0.78
0.61	0.70	0.53	0.42	0.28
	0.25	0.25	0.25	0.25

Asset Pricing:



Asset Pricing:



- **ZBDT prices are higher than BDT prices** in all the options.
- Before the crisis **the price difference is greater** than after the crisis.

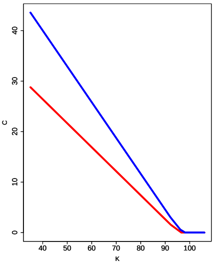
Second Case: Covid-19 Recession

Before the crisis: We analysed on February 14, 2020

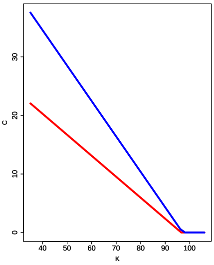
				6.05
			3.96	2.49
		2.73	1.72	1.03
	1.86	1.14	0.75	0.42
1.49	0.84	0.48	0.32	0.17
				5.77
			4.07	2.73
		3.08	1.93	1.29
	2.03	1.19	0.91	0.61
1.49	0.93	0.66	0.64	0.30
	0.25	0.25	0.25	0.25

Asset Pricing:

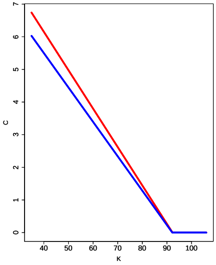
Knock-Up-Out



Knock-Double-Out



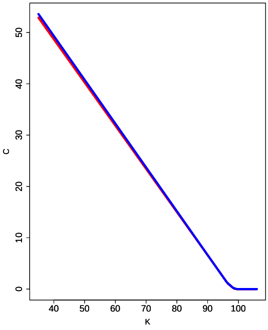
Knock-Down-In



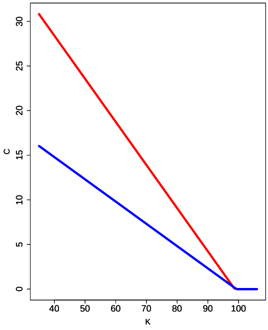
Second Case: Covid-19 Recession

Asset Pricing:

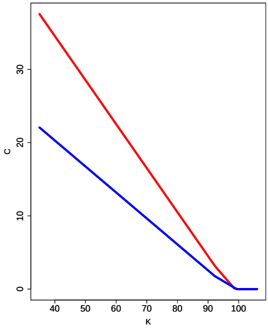
Knock-Down-Out



Knock-Up-In



Knock-Double-In



- It is possible to observe that in some cases the ZBDT model (blue line) is above the BDT price (red line), in others below and in one case are the same.

Conclusion

- We present a new interest rate model to value of financial derivatives.
- Its novelty is to add a new branch at each period to the classical BDT model that includes a small probability of falling into a recession (interest rate very close to zero).
- We value European call options in the 2008 recession and different barrier options in the Covid-19 recession in both models.
- We observe both small and large differences between both models.

Our Recommendation

Generate a bridge between new research results motivated by the financial crisis and classical literature on interest rate modeling.

THANK YOU FOR YOUR ATTENTION