Quantifying arbitrage

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Motivation

Going beyond the dichotomy arbitrage / no-arbitrage

In order to:

- consider model uncertainty without NA-restrictions
- avoid imposing too idealized constraints
- accommodate market shocks
- use when difficult to check NA condition, as in data driven models (at the same time, costly to impose martingale condition on pricing measures)

$\Rightarrow$ we suggest a way to quantify arbitrage and see what we can say when allowing for “small arbitrage” (pricing, hedging,...)

Setting and notations

- Discrete-time setting: $t = 0, 1, ..., T$
- S adapted process on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$: discounted asset price of d assets
- $H = (H_t)_{t=1}^T$ (predictable) trading strategies (denoted $H \in \mathcal{H}$)
- $|\,.\,|$: Euclidean norm on $\mathbb{R}^d$ (we can consider any p-norm, $p \in [1, \infty)$)
- $||X|| = \sum_t |X_t|$
- $\varepsilon > 0$

The notion of ε-arbitrage

Definition

A trading strategy $H \in \mathcal{H}$ is a strict ε-arbitrage if

$$\mathbb{P}\left[(H \bullet S)_T - \varepsilon||H|| \geq 0\right] = 1 \quad \text{and} \quad \mathbb{P}\left[... > 0\right] > 0.$$

- ε : "amount" of arbitrage
- $||H||$: "normalization" of strategies, to be able to talk of amount of arbitrage

(alternative interpretation of $\varepsilon||H||$: cost of managing the portfolio associated to H)

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Definition

A sequence $(H^k)_{k\in\mathcal{N}} \subseteq \mathcal{H}$ of trading strategies is called an ε-arbitrage if

$$\mathbb{P}\left[\liminf_{k\to\infty} \left\{(H^k \bullet S)_T - \varepsilon\|H^k\|\right\} \geq 0\right] = 1 \quad \text{and} \quad \mathbb{P}\left[\ldots > 0\right] > 0.$$

We write $\mathrm{NA}_\varepsilon(\mathbb{P})$ if the market admits no ε-arbitrage

Example

It is possible that:

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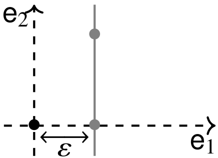
Example: $t = 0, 1,\ \Omega := \{\omega_1, \omega_2\},\ d = 2,\ S_0(\omega) := (0, 0),\ S_1(\omega) := \begin{cases} (\varepsilon, 0), & \omega = \omega_1 \\ (\varepsilon, 1), & \omega = \omega_2 \end{cases}$



$\Rightarrow\ H^k := (k, 1)$ is an ε-arbitrage

The notion of ε-martingale measure

Definition

A probability measure $\mathbb{Q} \sim \mathbb{P}$ is an ε-martingale measure if, for all t, $S_t \in L^1(\mathbb{Q})$ and

$$\left|\mathbb{E}_{\mathbb{Q}}[\Delta S_t|\mathcal{F}_{t-1}]\right| \leq \varepsilon.$$

Equivalently, for all $H \in \mathcal{H}$ with $||H|| \leq 1$,

$$\mathbb{E}_{\mathbb{Q}}[(H \bullet S)_T - \varepsilon||H||] \leq 0$$

(cf. ε-approximating martingale measures from Guo and Obloj)

$$\mathcal{M}_{\varepsilon}(\mathbb{P}) := \left\{\mathbb{Q}\ \varepsilon\text{-martingale measure s.t. } \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^{\infty}(\Omega, \mathcal{F}_T, \mathbb{P})\right\}$$

Relation between ε-arbitrage and ε-martingale measures

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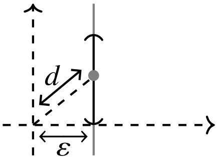

for all $\mathbb{Q} \sim \mathbb{P}$, $|\mathbb{E}_{\mathbb{Q}}[\Delta S_1|\mathcal{F}_0]| = d > \varepsilon$

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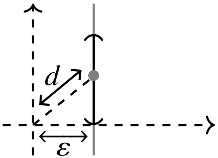

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Theorem (Quantitative FTAP)

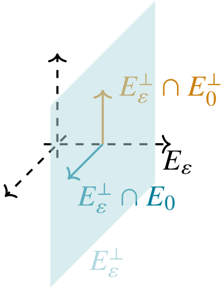
$$NA_{\varepsilon}(\mathbb{P}) \iff \mathcal{M}_{\varepsilon}(\mathbb{P}) \neq \emptyset$$

Canonical decomposition

Theorem

Assume absence of strict ε-arbitrage. Then any $H \in \mathcal{H}$ can be decomposed as

$$H = J + G + \tilde{G}, \qquad \text{with } J \in E_\varepsilon,\ G \in E_\varepsilon^\perp \cap E_0^\perp,\ \tilde{G} \in E_\varepsilon^\perp \cap E_0$$



$$E_\varepsilon = \{H \in \mathcal{H} : H_t \cdot \Delta S_t = \varepsilon |H_t| \quad \forall\ t\}$$

$$E_0 = \{H \in \mathcal{H} : H_t \cdot \Delta S_t = 0 \quad \forall\ t\}$$

The special role of $E_\varepsilon^\perp \cap E_0^\perp$

Theorem

$$NA_\varepsilon \iff \begin{cases} \textit{no strict } \varepsilon\textit{-arbitrage} \\ \textit{for any } G \in E_\varepsilon^\perp \cap E_0^\perp,\ (G \bullet S)_T \geq 0 \implies (G \bullet S)_T = 0 \end{cases}$$

That is, no strict ε arbitrage, and no classical arbitrage for a subfamily of strategies

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Theorem

The closure of $K := \{(H \bullet S)_T - \varepsilon\|H\| : H \in \mathcal{H}\}$ w.r.t. convergence in probability is

$$\overline{K} = \left\{(H \bullet S)_T - \varepsilon\|H\| + (G \bullet S)_T : H \in \mathcal{H}, G \in E_\varepsilon^\perp \cap E_0^\perp, H_t \cdot G_t = 0\ \forall t\right\}$$

Pricing-hedging duality

Theorem

Let $\Psi \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ be the payoff of a claim. Under $NA_\varepsilon(\mathbb{P})$ we have

$$\sup\left\{\mathbb{E}_{\mathbb{Q}}[\Psi] \colon \mathbb{Q} \in \mathcal{M}_\varepsilon(\mathbb{P})\right\} = \inf\Big\{x \colon \exists H \in \mathcal{H}, G \in E_\varepsilon^\perp \cap E_0^\perp \text{ s.t. } x + (H \bullet S)_T + (G \bullet S)_T \geq \Psi + \varepsilon\|H\|\Big\}$$

Interpretation: minimal cost to super-replicate and cover costs to manage portfolio H, while costs for G can be compensated with other trading strategies

ε-fair price

Definition

We say that $\psi \in \mathbb{R}$ is an ε-fair price for a claim Ψ if having Ψ at price ψ in the market does not introduce ε-arbitrage:

$$\liminf_{k\to\infty} \left\{(H^k \bullet S)_T + a^k(\Psi - \psi) - \epsilon \left(\|H^k\| + |a^k|\right)\right\} =: Y \geq 0 \implies Y = 0.$$

Theorem

Under $NA_\varepsilon(\mathbb{P})$ we have

$$\text{set of } \varepsilon\text{-fair prices for } \Psi = \bigcup_{\mathbb{Q}\in\mathcal{M}_\varepsilon(\mathbb{P})} \left[\mathbb{E}_\mathbb{Q}[\Psi] - \varepsilon, \mathbb{E}_\mathbb{Q}[\Psi] + \varepsilon\right]$$

The critical value

Definition (critical value)

$$\varepsilon(\mathbb{P}) := \inf\left\{\varepsilon > 0 \colon \text{there is no (strict) } \varepsilon\text{-arbitrage under } \mathbb{P}\right\}$$

Equivalently,

$$\varepsilon(\mathbb{P}) = \inf\left\{\varepsilon \geq 0 : \mathcal{M}_{\varepsilon}(\mathbb{P}) \neq \emptyset\right\} = \sup_{H \in \mathcal{H}} \operatorname{ess\,inf}_{\mathbb{P}} \frac{(H \bullet S)_T}{||H||} \chi_{\{||H|| \neq 0\}}$$

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Theorem

If $\varepsilon > \varepsilon(\mathbb{P})$, then

- $NA_\varepsilon(\mathbb{P})$ *holds*
- $K = \overline{K}$
- $\sup\left\{\mathbb{E}_{\mathbb{Q}}[\Psi]\colon \mathbb{Q} \in \mathcal{M}_\varepsilon(\mathbb{P})\right\} = \inf\{x\colon \exists H \in \mathcal{H} \text{ s.t. } x + (H \bullet S)_T \geq \Psi + \varepsilon\|H\|\}$

Stability

Definition

We define the adapted L^∞-distance as

$$\mathcal{AW}_\infty(\mathbb{P}, \mathbb{P}') := \inf \left\{ \operatorname{ess\,sup}_\pi \|\Delta X - \Delta Y\| : \pi \in \Pi(\mu, \nu),\ \pi \text{ bicausal} \right\}$$

Theorem

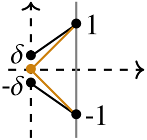
- $|\varepsilon(\mathbb{P}) - \varepsilon(\mathbb{P}')| \leq \mathcal{AW}_\infty(\mathbb{P}, \mathbb{P}')$
- $\mathcal{M}_\varepsilon(\mathbb{P}) \neq \emptyset \implies \mathcal{M}_{\varepsilon + \mathcal{AW}_\infty(\mathbb{P}, \mathbb{P}')}(\mathbb{P}') \neq \emptyset$

In particular, if $\mathbb{P}$ satisfies classical NA ($\Rightarrow \varepsilon(\mathbb{P}) = 0$), then for $\mathbb{P}'$ in a δ-neighborhood we can have at most δ arbitrage

Stability

To have the above stability results, in the definition of our distance:

- we cannot drop adaptedness:



- $\mathcal{AW}_\infty(\mathbb{P}_\delta, \mathbb{P}) = 2, \quad \mathcal{W}_\infty(\mathbb{P}_\delta, \mathbb{P}) = 2\delta$
- $\mathcal{M}_0(\mathbb{P}) \neq \emptyset, \quad \mathcal{M}_\varepsilon(\mathbb{P}_\delta) = \emptyset \ \ \forall\, \varepsilon < 1-\delta$
- $\varepsilon(\mathbb{P}) = 0, \quad \varepsilon(\mathbb{P}_\delta) = 1-\delta$

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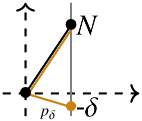

$p_\delta = \delta/(M+\delta)^2$

- $\mathcal{AW}_\infty(\mathbb{P}_N, \mathbb{P}) = N + \delta, \quad \mathcal{AW}_2(\mathbb{P}_N, \mathbb{P}) = \sqrt{\delta}$
- $\mathcal{M}_0(\mathbb{P}) \neq \emptyset, \quad \mathcal{M}_\varepsilon(\mathbb{P}_N) = \emptyset \ \ \forall\, \varepsilon < N$
- $\varepsilon(\mathbb{P}) = 0, \quad \varepsilon(\mathbb{P}_N) = N$

Conclusions

In the presented paper, we:

- introduced quantification of concept of arbitrage
- made sense of pricing and hedging under small arbitrage
- established q-FTAP and q-duality
- proved stability w.r.t. new $\mathcal{AW}_\infty$-distance

Thank you for your attention!