Equivalent formulations of optimal control problems with maximum cost and applications

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Outline

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1. Motivation
   - The covid problem
2. Towards a general model
   - Formulations with constraints
     - Reformulation $\mathcal{P}_0$
     - Reformulation $\mathcal{P}_1$
     - Reformulation $\mathcal{P}_2$
   - Formulations without state constraints
     - Reformulation $\mathcal{P}_3$
     - Reformulation $\mathcal{P}_3^\theta$
3. Numerical examples
   - Application to the classical SIR model
4. Conclusion and future work

Outline

Context: Covid desease

High peaks overcrowd the healthy system.

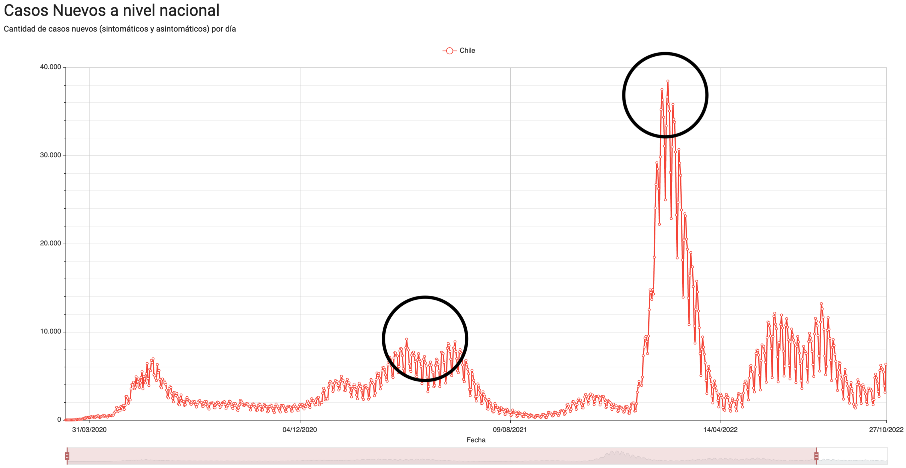

Figure: Chile's data from CMM Covid-19 Visualization:
`https://covid-19vis.cmm.uchile.cl/`

SIR model

A classical SIR model corresponds to:

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where:

- S: portion of susceptible individuals.
- I: portion of infected individuals.
- R: portion of recovered individuals.
- β: transmission rate.
- γ: recovery rate.

And

$$S + I + R = 1.$$

Example

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They worked with a SIR model minimizing $\max_{t\in[0,T]} I(t)$.

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Interventions $b(t)$ are modeled as a factor in rate transmission which take place in a interval $[t_i, t_i + \tau]$.

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Interventions $b(t)$ are modeled as a factor in rate transmission which take place in a interval $[t_i, t_i + \tau]$.
Remark: They didn't use any optimal control tool in their work.

Outline

Formulation general problem

We consider the following dynamical system in a domain $\mathcal{D} \subset \mathbb{R}^{n+1}$.

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \end{cases} \qquad \text{(Dynamics)}$$

$\mathcal{U} := \{u(\cdot) : [0, T] \mapsto U, \text{mesurable}\}$ and $(x_0, y_0) \in \mathcal{D},\ T > 0$.

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The solutions set:

$$\mathcal{S} := \{(x(\cdot), y(\cdot)) \in \mathcal{AC}([0, T], \mathbb{R}^{n+1}), \quad \text{sol. of (Dynamics) for } u(\cdot) \in \mathcal{U} \text{ with } (x(0), y(0)) = (x_0, y_0)\}$$

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The optimal control problem:

$$\mathcal{P} : \quad \inf_{u(\cdot) \in \mathcal{U}} \left(\max_{t \in [0,T]} y(t) \right) = \inf_{(x(\cdot), y(\cdot)) \in \mathcal{S}} \left(\max_{t \in [0,T]} y(t) \right)$$

State of art

- L^∞-criterion.

$$\inf_{u(\cdot)} \operatorname{ess\,sup}_{t\in[t_0,T]} y(t)$$

where $y(t) = \eta(\xi(t))$ with $\xi(\cdot)$ solution of a controlled system $\dot{\xi} = \phi(\xi, u)$, $\xi(t_0) = \xi_0$.

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- Typically

$$\min\left(\partial_t V + \inf_u \partial_\xi V.\phi(x,u)\ ,\ V - \eta\right) = 0\ .$$

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- Typically

$$\min\left(\partial_t V + \inf_u \partial_\xi V.\phi(x,u)\ ,\ V - \eta\right) = 0\ .$$

- There is no practical tools to solve such problems, to the best of our knowledge.

Optimal control problems

Our objective: use a more classical Mayer, Lagrange or Bolza formulation.

$$\min \quad g(x(T)) + \int_0^T f^0(t, x(t), u(t))dt$$

$$\dot{x} = f(t, x(t), u(t)) \qquad\qquad t \in [0, T]$$

$$u \in \mathcal{U},\ x(0) \in M_0,\ x(T) \in M_1$$

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State constraints:

$$x(t) \in A, \quad t \in [0, T]$$

Outline

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Reformulation $\mathcal{P}_0$

The first basic reformulation is

$$\mathcal{P}_0 : \inf_{u(\cdot)\in\mathcal{U}} z(T)$$

for the extended dynamics in $\mathcal{D} \times \mathbb{R}$

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = 0 \end{cases}$$

under the state constraint

$$\mathcal{C} : \quad z(t) - y(t) \geq 0, \; t \in [0, T]$$

where $(x(0), y(0)) = (x_0, y_0)$ and $z(0)$ is free.

Outline

Reformulation $\mathcal{P}_1$

The dynamic in $\mathcal{D} \times \mathbb{R}$:

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = \max(g(x, y, u), 0)(1 - v) \quad , v \in [0, 1] \end{cases} \tag{$\mathrm{Dyn}_{z,v}$}$$

the optimal control problem:

$$\mathcal{P}_1 : \quad \inf_{(u(\cdot), v(\cdot))} z(T)$$

under the constraint:

$$\mathcal{C} : \quad z(t) \geq y(t), \; t \in [0, T]$$

where $x(\cdot), y(\cdot), z(\cdot)$ is solution of $(\mathrm{Dyn}_{z,v})$ with $x(0) = x_0$, $y(0) = y_0$, $z(0) = y_0$.

Idea

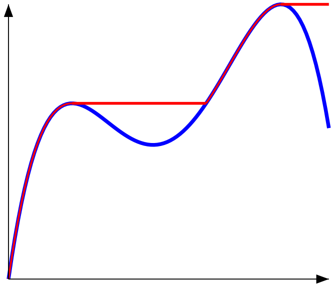

Figure: Illustration of the function z (red) corresponding to a function y (blue)

Equivalence

Hypotheses

1. *U is a compact set.*
2. *The maps f and g are C^1 on $\mathcal{D} \times U$.*
3. *The maps f and g have linear growth, that is there exists a number $C > 0$ such that*

$$||f(x,y,u)|| + |g(x,y,u)| \leq C(1 + ||x|| + |y|),\ (x,y) \in \mathcal{D},\ u \in U$$

Equivalence

Theorem

For any control $u(\cdot) \in \mathcal{U}$, the optimal control problem

$$\inf_{v \in \mathcal{V}} z(T) \text{ under the constraint } \mathcal{C} \tag{1}$$

admits an optimal solution. Moreover, an optimal solution verifies

$$z(T) = \max_{t \in [0,T]} y(t). \tag{2}$$

and is reached for a control $v(\cdot)$ that takes values in $\{0, 1\}$.

Equivalence

Theorem

If $(u^\star(\cdot), v^\star(\cdot))$ *is optimal for* $\mathcal{P}_1$*, then* $u^\star(\cdot)$ *is optimal for* $\mathcal{P}$*.*

Conversely, if $u^\star(\cdot)$ *is optimal for* $\mathcal{P}$*, then* $(u^\star(\cdot), v^\star(\cdot))$ *is optimal for* $\mathcal{P}_1$ *where* $v^\star(\cdot)$ *is optimal for the problem* (1) *fixing* $u^\star(\cdot)$*.*

Outline

Reformulation $\mathcal{P}_2$

The extended dynamics in $\mathcal{D} \times \mathbb{R}$.

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = \max(g(x, y, u), 0)(1 - v) \quad , v \in [0, 1] \end{cases} \tag{$\text{Dyn}_{z,v}$}$$

the optimal control problem

$$\mathcal{P}_2 : \quad \inf_{(u(\cdot), v(\cdot))} z(T)$$

under the constraint

$$\mathcal{C}_m : \quad \max(y(t) - z(t), 0)(1 - v(t)) + z(t) - y(t) \geq 0, \quad \text{a.e. } t \in [0, T]$$

where $x(\cdot), y(\cdot), z(\cdot)$ is solution of $(\text{Dyn}_{z,v})$ with $x(0) = x_0$, $y(0) = y_0$, $z(0) = y_0$.

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Reformulation $\mathcal{P}_3$

We posit $\Pi = (x, y, z) \in \mathcal{D} \times \mathbb{R}$ with dynamic:

$$\dot{\Pi} \in F(\Pi) := \bigcup_{(u,v)\in U\times[0,1]} \begin{bmatrix} f(x,y,u) \\ g(x,y,u) \\ h(x,y,z,u,v) \end{bmatrix} \tag{3}$$

and

$$h(x,y,z,u,v) = \max(g(x,y,u),0)(1 - v\mathbb{1}_{\mathbb{R}^+}(z-y)).$$

Let $\mathcal{S}_\ell := \{\Pi(\cdot) \in AC., \dot{\Pi} \in F(\Pi)$ and $\Pi(0) = (x_0, y_0, y_0)$

$$\mathcal{P}_3 : \quad \inf_{\Pi(\cdot)\in\mathcal{S}_\ell} z(T).$$

Equivalence

Hypotheses

$$\forall (x,y) \in \mathcal{D}, \quad G(x,y) := \bigcup_{u \in U} \begin{bmatrix} f(x,y,u) \\ g(x,y,u) \end{bmatrix} \text{ is convex,}$$

Equivalence

Hypotheses

$$\forall (x, y) \in \mathcal{D}, \quad G(x, y) := \bigcup_{u \in U} \begin{bmatrix} f(x, y, u) \\ g(x, y, u) \end{bmatrix} \text{ is convex,}$$

Proposition 1

$\mathcal{P}_3$ admits an optimal solution. Moreover, any optimal solution $\Pi(\cdot) = (x(\cdot), y(\cdot), z(\cdot))$ verifies

$$z(T) = \max_{t \in [0, T]} y(t)$$

with $(x(\cdot), y(\cdot))$ solution of (Dynamics) *for some control $u(\cdot) \in \mathcal{U}$ that is optimal for $\mathcal{P}$.*

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Reformulation $\mathcal{P}_3^\theta$

A dynamic parameterized by $\theta > 0$

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = h_\theta(x, y, z, u, v) \end{cases} \tag{4}$$

with

$$h_\theta(x, y, z, u, v) = \max(g(x, y, u), 0)(1 - v\, e^{-\theta \max(y - z, 0)})$$

The family of Mayer problems

$$\mathcal{P}_3^\theta : \quad \inf_{\Pi(\cdot) \in \mathcal{S}_\theta} z(T)$$

where $\mathcal{S}_\theta$ denotes the set of absolutely continuous solutions $\Pi(\cdot) = (x(\cdot), y(\cdot), z(\cdot))$ of (4) for the initial condition $\Pi(0) = (x_0, y_0, y_0)$

Approximation

Proposition 2

For any increasing sequence $\theta_n \to +\infty$, the problem $\mathcal{P}_3^{\theta_n}$ admits an optimal solution, and for the optimal solutions $(x_n(\cdot), y_n(\cdot), z_n)(\cdot))$ of $\mathcal{P}_3^{\theta_n}$,

- *$(x_n(\cdot), y_n(\cdot))$ converges, up to sub-sequence, uniformly to an optimal solution $(x^\star(\cdot), y^\star(\cdot))$ of $\mathcal{P}$.*
- *its derivatives weakly to $(\dot{x}^\star(\cdot), \dot{y}^\star(\cdot))$ in L^2.*
- $z_n(T) \nearrow \max_{t\in[0,T]} y^\star(t)$.

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SIR model

Recall the SIR dynamic

$$\begin{aligned}\dot{S}(t) =& -(1-u(t))\beta S(t)I(t)\\ \dot{I}(t) =& (1-u(t))\beta S(t)I(t) - \gamma I(t)\\ \dot{R}(t) =& \gamma I(t)\end{aligned}$$

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We add the constraint

$$\int_0^T u(t)dt \leq Q$$

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We add the constraint

$$\int_0^T u(t)dt \leq Q$$

And we want

$$\min_u \max_{t\in[0,T]} I(t)$$

Analytical solution

We proved that for an initial conditions $I_0 = I(0) > 0$ and $S_0 = S(0) > S_h = \mathcal{R}_0^{-1} = \gamma/\beta$, the optimal solution is the feedback control

$$\psi(I, S) := \begin{cases} 1 - \frac{S_h}{S} & \text{if } I = \bar{I} \text{ and } \; S > S_h \\ 0 & \text{otherwise} \end{cases}$$

where

$$\bar{I} := \frac{I_0 + S_0 - S_h - S_h \log\left(\frac{S_0}{S_h}\right)}{Q\beta S_h + 1}$$

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For numerical examples:

β	γ	T	Q	$S(0)$	$I(0)$	$\bar{I}$
0.21	0.07	300	28	$1 - 10^{-6}$	10^{-6}	0.1015

Analytical solution

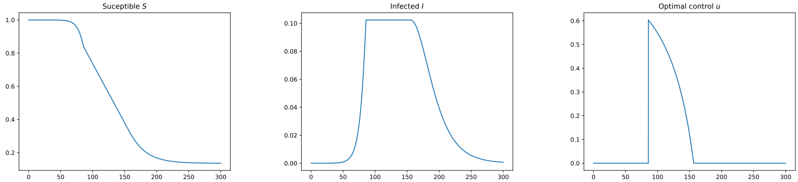

Figure: The optimal solution for the SIR problem

Numerical solutions

To improve convergence we used the approximation:

$$\frac{\log\left(e^{\lambda\xi}+1\right)}{\lambda} \underset{\lambda\to+\infty}{\longrightarrow} \max(\xi,0), \quad \xi\in\mathbb{R}$$

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Using $\lambda = 100$ we obtain

problem	$\max_{t\in[0,T]} y(t)$	computation time
$\mathcal{P}_0$	0.1015	10 s
$\mathcal{P}_1$	0.1015	12 s
$\mathcal{P}_2$	0.1015	13 s

Table: Comparison of performances for problems $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$

Numerical solutions

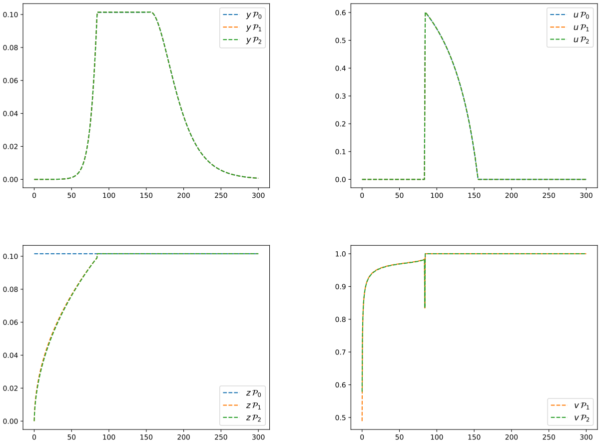

Figure: Comparisons of numerical results for the methods $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$

Numerical solutions

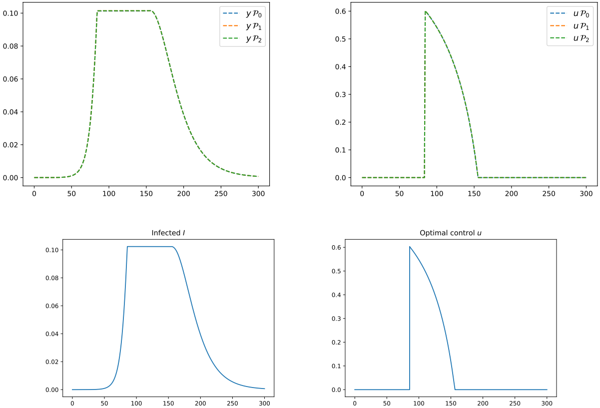

Figure: Comparision for the SIR problem

Outline

Conclusion and ongoing work

Formulation	$\mathcal{P}_0$	$\mathcal{P}_1$ or $\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_3^\theta$
suitable to direct methods	yes	yes	no	yes
suitable to HJB methods	no	yes	yes	yes
suitable to shooting methods	no	no	no	yes
provides approximations from below	no	no	no	yes

Table: Comparison of the different formulations

Conclusion and ongoing work

- We have proposed two kinds of formulations: one with **state or mixed constraints** and another one **without any constraint**.

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- For the latter one, we have proposed an approximation scheme generated. Although this second approach requires larger computation time, it complements the first ones providing **approximations** of the optimal value **from above**.

Conclusion and ongoing work

- We have proposed two kinds of formulations: one with **state or mixed constraints** and another one **without any constraint**.
- For the latter one, we have proposed an approximation scheme generated. Although this second approach requires larger computation time, it complements the first ones providing **approximations** of the optimal value **from above**.
- The study of necessary optimality conditions will be the matter of a future work.

Thanks!

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Received: 10 March 2022 / Accepted: 7 August 2022

Assumptions

Hypotheses

1. *U is a compact set.*
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3. *The maps f and g have linear growth, that is there exists a number $C > 0$ such that*

$$||f(x,y,u)|| + |g(x,y,u)| \leq C(1 + ||x|| + |y|), \; (x,y) \in \mathcal{D}, \; u \in U$$

Sketch of proof

Consider the augmented dynamics

$$\dot{\Pi} \in F^{\dagger}(\Pi) := \bigcup_{(u,v,\alpha)\in U\times[0,1]^2} \begin{bmatrix} f(x,y,u) \\ g(x,y,u) \\ h^{\dagger}(x,y,z,u,v,\alpha) \end{bmatrix} \tag{5}$$

with

$$h^{\dagger}(x,y,z,u,v,\alpha) = (1-\alpha)h(x,y,z,u,v) + \alpha \max_{w\in U} h(x,y,z,w,0)$$

$\implies \Pi^{\star}(\cdot) = (x^{\star}(\cdot), y^{\star}(\cdot), z^{\star}(\cdot))$ of (5) optimal

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$\implies \Pi^{\star}(\cdot) = (x^{\star}(\cdot), y^{\star}(\cdot), z^{\star}(\cdot))$ of (5) optimal

Any admissible solution $(x(\cdot), y(\cdot), z(\cdot))$ of $\mathcal{P}_1$ belongs to $\mathcal{S}_\ell \subset \mathcal{S}_\ell^{\dagger}$. Then

$$z^{\star}(T) \leq \inf\{z(T);\ (x(\cdot), y(\cdot), z(\cdot)) \text{ sol. of } (\mathrm{Dyn}_{z,v}) \text{ with } \mathcal{C}\}. \tag{6}$$

Sketch of proof

Besides, any solution $\Pi(\cdot) = (x(\cdot), y(\cdot), z(\cdot))$ in $\mathcal{S}_\ell^\dagger$ verifies $z(t) \geq y(t)$

$$z(T) \geq \max_{t \in [0,T]} y(t) \tag{7}$$

Sketch of proof

Besides, any solution $\Pi(\cdot) = (x(\cdot), y(\cdot), z(\cdot))$ in $\mathcal{S}_\ell^\dagger$ verifies $z(t) \geq y(t)$

$$z(T) \geq \max_{t\in[0,T]} y(t) \tag{7}$$

Thanks to Assumptions 1 and 2, we obtain

$$z^\star(T) \geq \max_{t\in[0,T]} y^\star(t) \geq \inf_{u\in\mathcal{U}} \left\{ \max_{t\in[0,T]} y(t);\ (x(\cdot), y(\cdot)) \text{ sol. of (Dynamics)} \right\} \tag{8}$$

where $(x^\star(\cdot), y^\star(\cdot))$ is solution of (Dynamics) for a certain $u^\star(\cdot) \in \mathcal{U}$.

Approximation by norm L^p

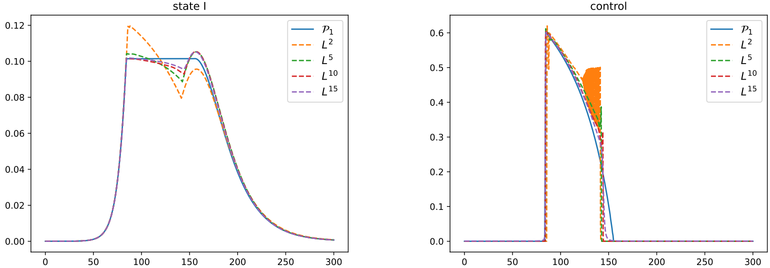

Figure: Numerical solutions for problems $\mathcal{P}_{L^p}$

Approximation by norm L^p

p	$\max_{t\in[0,T]} y(t)$	$\|\|y(t)\|\|_p$	computation time
2	0.119653	1.0222	34 s
5	0.105244	0.2474	14 s
10	0.105375	0.15678	13 s
15	0.105170	0.13549	17 s

Table: Comparison of the numerical results with the L^p approximation