Diffusion in inverse problems and inverse problems in diffusion

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Tentative road map

1. The inpainting problem
   - T1, TV and mixed weighted T1-TV inpainting
   - Curvature-driven diffusion inpainting
   - A two step CDD + T1-TV inpainting method
   - Numerical implementation and results
2. An inverse heat conduction problem
   - Origins of the problem
   - A brief historical mathematical tracking of the problem
   - Calderón's problem
   - Inverting the conductivity-to-temperature mapping
   - The inverse problem
   - Regularization: finding the right penalizers
   - Examples and numerical experiments
   - Open problems

The inpainting problem

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- $u : \Omega \to [0, 1]$ is the light intensity function.

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- $D \subset \Omega$ is the region missing data. (Red in the figure)
- $u : \Omega \to [0, 1]$ is the light intensity function.
- $v = u \,|_{\Omega \setminus D}$ is the (possibly noisy) known part of u.

Tikhonov-Phillips Inpainting Method

The order 1 Tikhonov-Phillips inpainting is obtained by minimizing

$$\mathcal{J}(u) = \|\mathcal{T}u - v\|_{L^2}^2 + \lambda\|\nabla u\|_{L^2}^2, \quad \text{(T1I)}$$

with respect to u.

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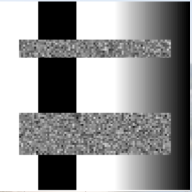

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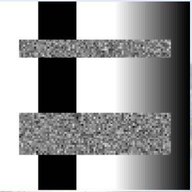

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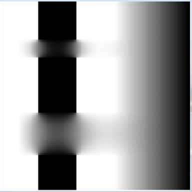

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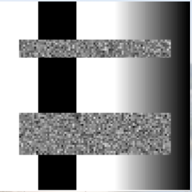

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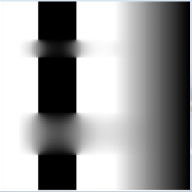

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To enhance edge preservation, add an anisotropy inducing matrix A to (T1I),

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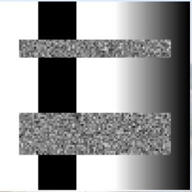

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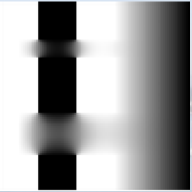

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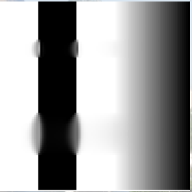

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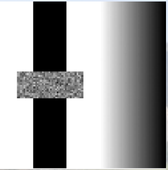

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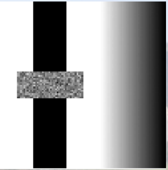

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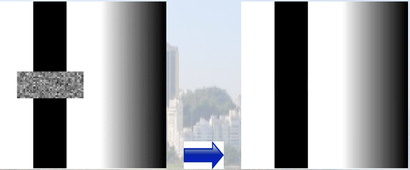

Occluded image and TVI inpainting for a **thin** occlusion.

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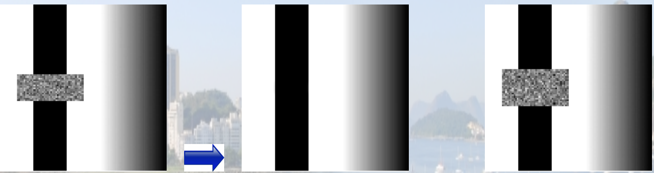

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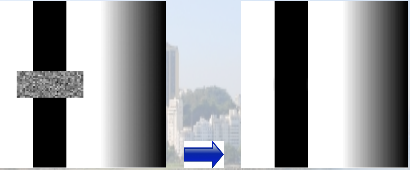

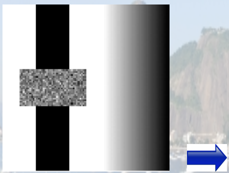

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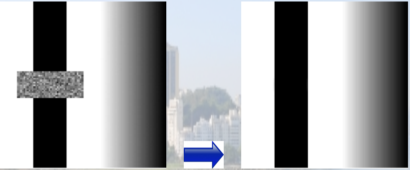

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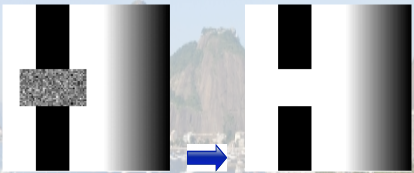

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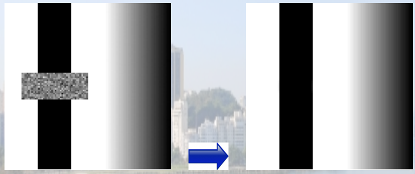

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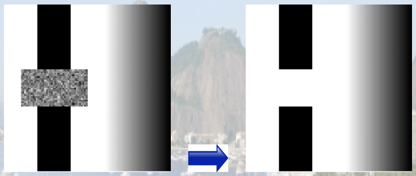

Occluded image and TVI inpainting for a **wide** occlusion.

Euler-Lagrange equation of (TVI) $\Rightarrow$ inside the occlusion, the minimizer of (TVI) is the steady state solution of

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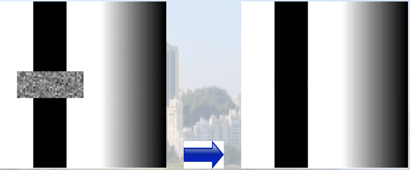

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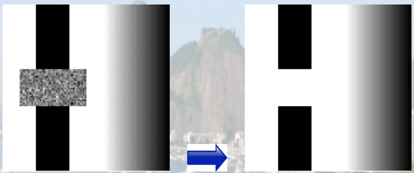

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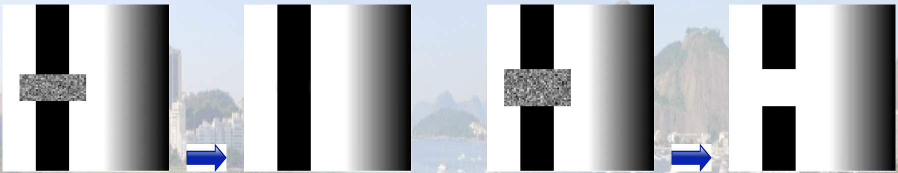

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[1] Chan, T.F. and Shen, J. *Mathematical models for local nontexture inpaintings, SIAM J. Appl. Math., 2002*

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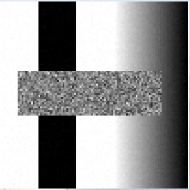

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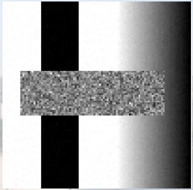

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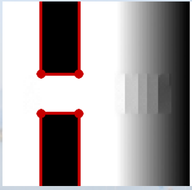

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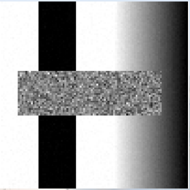

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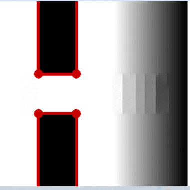

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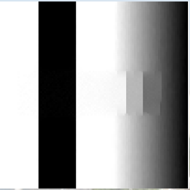

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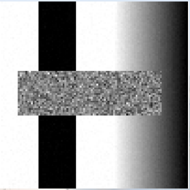

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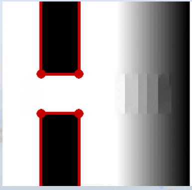

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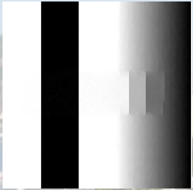

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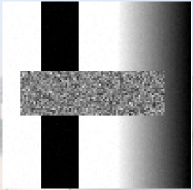

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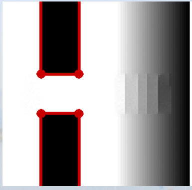

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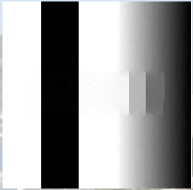

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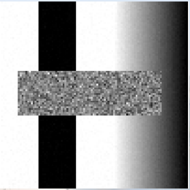

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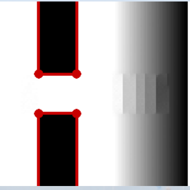

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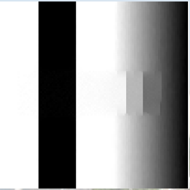

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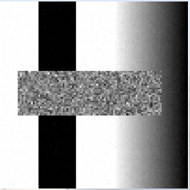

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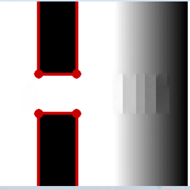

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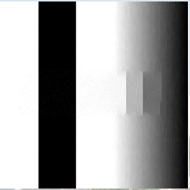

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- Object connectivity.

Mixed Weighted Regularization Inpainting

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$\theta : \Omega \to [0,1]$ is a spatially varying weighting function.

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(Mazzieri-Spies-Temperini, "Mixed spatially varying $L^2 - BV$ regularization of inverse ill-posed problems", Journal of Inverse and Ill-Posed Problems, 2015: 23(6):571-585.)

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A new two-step inpainting method

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v

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1. Set $u^{(0)} \in \mathbb{R}^{M^2}$, $n = 0$ and $f \doteq \nabla \cdot \left[\frac{|\kappa|}{|\nabla u|} \nabla u\right]$, where $\kappa = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right]$

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2. Compute a step of the Adams-Bashforth-Moulton predictor-corrector method:

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Convolution $u_p = G * u_p^*$, where G is a low-variance Gaussian kernel.

Anisotropy matrix field and weighting function

Anisotropy matrix field A

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We want $\theta \approx 0$ where $|\nabla u_p|$ is small and $\theta \approx 1$ where $|\nabla u_p|$ is large.

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The two-step inpainting algorithm

Full algorithm

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Full algorithm

- Find u_p^* by solving $\dfrac{\partial u}{\partial t} = \nabla \cdot \left[\dfrac{|\kappa|\chi_D + \chi_{\Omega \setminus D}}{|\nabla u|} \nabla u \right] + \dfrac{1}{\lambda}(u - v)\chi_{\Omega \setminus D}.$

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[1] Ibarrola, F. and Spies, R. *Image restoration with a half-quadratic approach to mixed*

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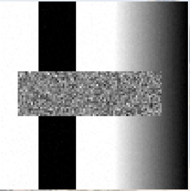

Occluded Image.

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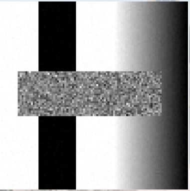

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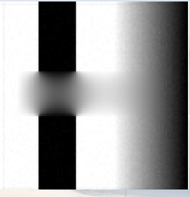

T1I inpainting.

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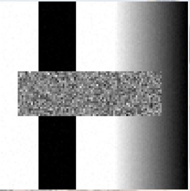

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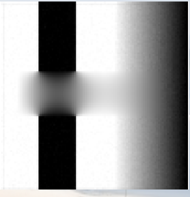

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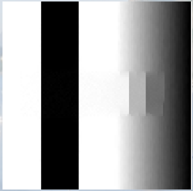

CDD inpainting.

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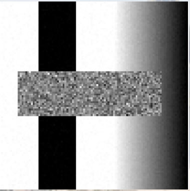
$$\mathcal{J}(u) = \|\mathcal{T}u - v\|_{L^2}^2 + \lambda_{T1}\||\sqrt{1-\theta}\, A\nabla u|\|_{L^2}^2 + \lambda_{TV}\||\theta A \nabla u|\|_{L^1}.$$



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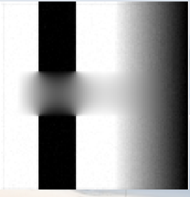
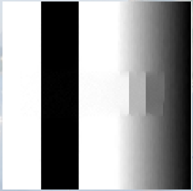
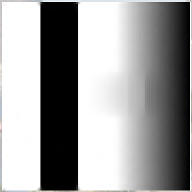


T1I inpainting.



CDD inpainting.



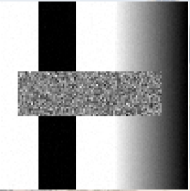
Two-step CDD + T1A-TVA inpainting.

The two-step inpainting algorithm

Full algorithm

- Find u_p^* by solving $\dfrac{\partial u}{\partial t} = \nabla \cdot \left[\dfrac{|\kappa|\chi_D + \chi_{\Omega\setminus D}}{|\nabla u|} \nabla u \right] + \dfrac{1}{\lambda}(u - v)\chi_{\Omega\setminus D}$.
- Convolve $u_p = G * u_p^*$, and compute $A = A(\nabla u_p)$ and $\theta = \theta(\nabla u_p)$.
- Find the minimizer of

$$\mathcal{J}(u) = \|\mathcal{T}u - v\|_{L^2}^2 + \lambda_{T1}\| |\sqrt{1-\theta}\, A\nabla u| \|_{L^2}^2 + \lambda_{TV}\| |\theta A \nabla u| \|_{L^1}.$$



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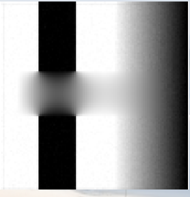

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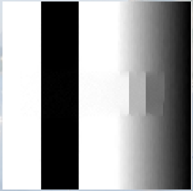

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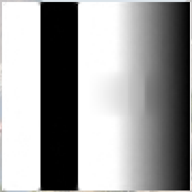

Two-step CDD + T1A-TVA inpainting.

	T1I	CDD	T1A-TVA
$PSNR = 20\log_{10}\left(M\|u_0 - \hat{u}\|^{-1}\right)$	20.140	35.496	36.330

Diffusion methods in inpainting

Inpainting as an inverse problem: Total Variation + Curvature Driven Diffusion Method

Diffusion methods in inpainting

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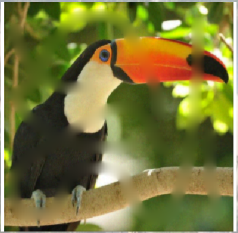

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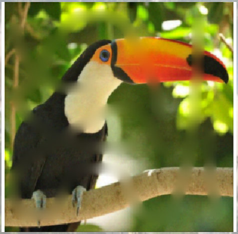

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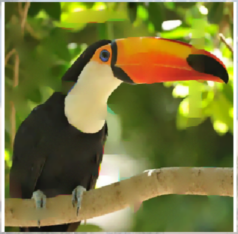

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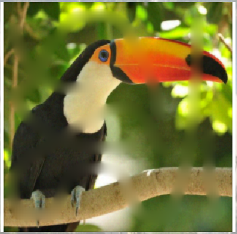

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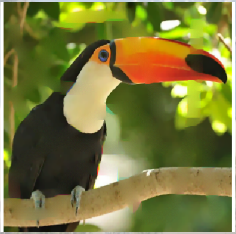

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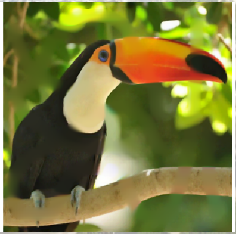


T1A-TVA inpainting.

Diffusion methods in inpainting

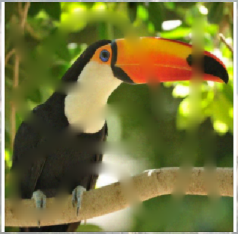
Inpainting as an inverse problem: Total Variation + Curvature Driven Diffusion Method

Occluded Image.

Performance comparisons

Gaussian additive noise= 2 %

	T1I	CDD	T1A-TVA
PSNR	20.815	22.292	22.551



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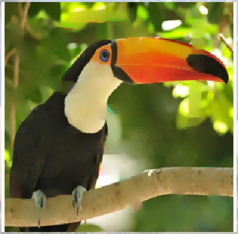

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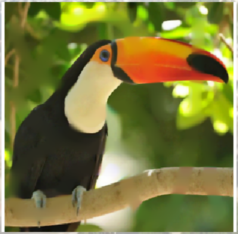

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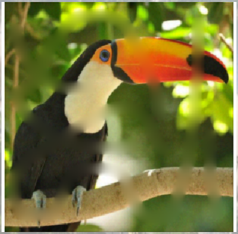
F. Ibarrola, R. Spies, "*A two-step mixed inpainting method with curvature-based anisotropy and spatial adaptivity*´´, Inverse Problems and Imaging, Volume 11, No. 2, 2017, pp 247-262,[7]



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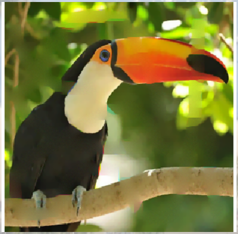

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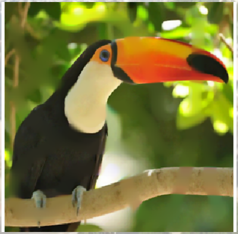

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Identification of the conductivity in a heat conduction problem

A recent result ... **(Int. J. Numer. Methods Eng. 2022; 1-18.) First published: 02 November 2022**

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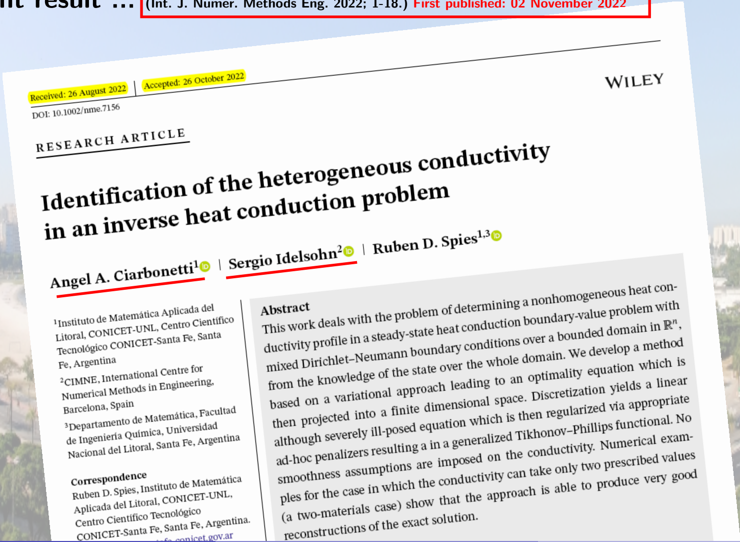

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RESEARCH ARTICLE

Identification of the heterogeneous conductivity in an inverse heat conduction problem

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[2]CIMNE, International Centre for Numerical Methods in Engineering, Barcelona, Spain

[3]Departamento de Matemática, Facultad de Ingeniería Química, Universidad Nacional del Litoral, Santa Fe, Argentina

Correspondence
Ruben D. Spies, Instituto de Matemática Aplicada del Litoral, CONICET-UNL, Centro Científico Tecnológico CONICET-Santa Fe, Santa Fe, Argentina.
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Abstract

This work deals with the problem of determining a nonhomogeneous heat conductivity profile in a steady-state heat conduction boundary-value problem with mixed Dirichlet–Neumann boundary conditions over a bounded domain in $\mathbb{R}^n$, from the knowledge of the state over the whole domain. We develop a method based on a variational approach leading to an optimality equation which is then projected into a finite dimensional space. Discretization yields a linear although severely ill-posed equation which is then regularized via appropriate ad-hoc penalizers resulting a in a generalized Tikhonov–Phillips functional. No smoothness assumptions are imposed on the conductivity. Numerical examples for the case in which the conductivity can take only two prescribed values (a two-materials case) show that the approach is able to produce very good reconstructions of the exact solution.

Origin of the problem

Flux manipulation - design of thermal materials or metamaterials

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Optimization-based design of easy-to-make devices for heat flux manipulation

Check for updates

Víctor D. Fachinotti[a,*], Ángel A. Ciarbonetti[a], Ignacio Peralta[a], Ignacio Rintoul[b]

[a] *Centro de Investigación de Métodos Computacionales (CIMEC), Universidad Nacional del Litoral (UNL)/ Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Predio CONICET "Dr. Alberto Cassano", Colectora Ruta Nac. 168 km 0, Paraje El Pozo, CP 3000, Santa Fe, Argentina*

[b] *Instituto de Desarrollo Tecnológico para la Industria Química (INTEC), Universidad Nacional del Litoral (UNL)/ Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Predio CONICET "Dr. Alberto Cassano", Colectora Ruta Nac. 168 km 0, Paraje El Pozo, CP 3000, Santa Fe, Argentina*

A R T I C L E I N F O

Keywords:
Heat flux manipulation
Optimization-based design
Easy-to-make device
Heat flux inversion
Metamaterial

A B S T R A C T

In this work, we present a new method for the design of heat flux manipulating devices, with emphasis on their manufacturability. The design is obtained as solution of a nonlinear optimization problem where the objective function represents the given heat flux manipulation task, and the design variables define the material distribution in the device. In order to facilitate the fabrication of the device, the material at a given point is chosen between two materials with highly different conductivity. By this way, the whole device can be seen, in the large scale, as a metamaterial having a specific anisotropic effective conductivity. As an application example, we designed a heat flux inverter which was so simple that it could be hand-made. The performance of this device for heat flux inversion was experimentally tested, proving that it was more efficient than a more complex device designed using the classical transformation thermodynamics approach.

Flux reversal (a way of violating Fourier's law ?)

Numerical results ...

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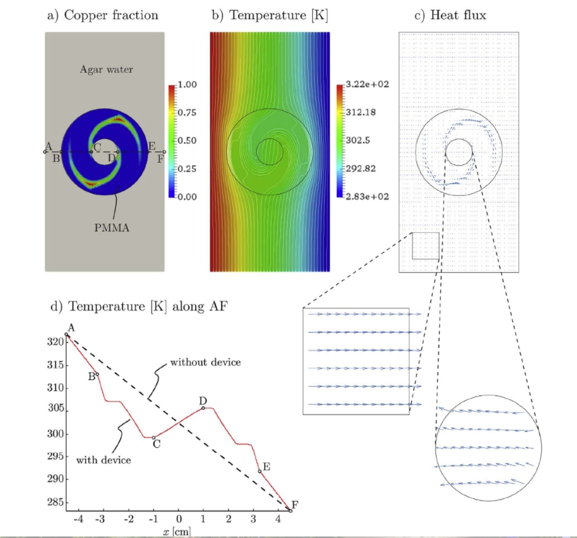

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Experimental set-up and validation ...

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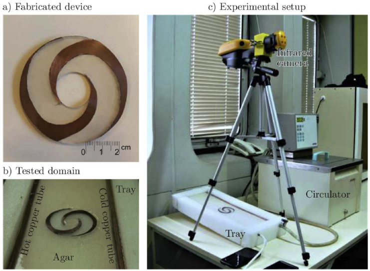

Fig. 7. Experimental heat flux inversion: a) hand-made device; b) domain of analysis; c) experimental setup.

1 Experimental set-up

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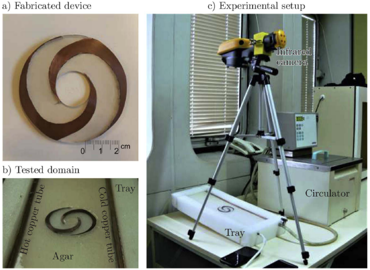
Experimental set-up and validation ...



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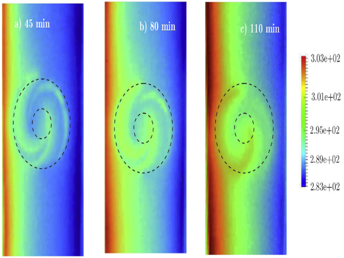

Fig. 8. Thermal images of the agar plate and the device from the infrared camera: a) 15 min; b) 45 min; and c) 110 min.

1 Experimental set-up

2 Experimental results

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Comparison...

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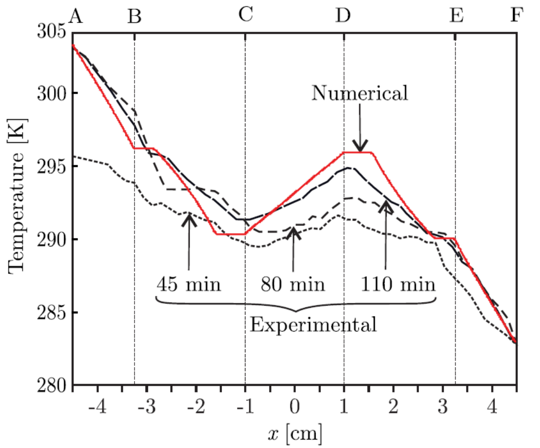

Fig. 9. Temperature along the line AF: experimental vs. numerical.

A brief historical mathematical tracking of the problem

Problem: of determining the elliptic coefficient profile function in an elliptic boundary value problem.

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Applications: electrical conductivity problems, oil resevoir and ground water flow problems ([3], [4], [8], [11], [17])

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A. P. Calderón's problem ([6], 1980, ATAS of SBM, Rio de Janeiro 1980):
$\Omega \subset \mathbb{R}^n$, $n \geq 2$ bounded, $\partial\Omega$ Lipschitz, $u_\varphi \in H^1(\Omega)$ solution of the Dirichlet BVP

$$\begin{cases} \nabla.(k\nabla u) = 0, & x \in \Omega, \\ u = \varphi, & x \in \partial\Omega, \end{cases}$$

and

$$Q_k(\varphi) = \int_\Omega k(x)\left(\nabla u_\varphi(x)\right)^2\, dx = \int_{\partial\Omega} \varphi(x)k(x)\frac{\partial u_\varphi}{\partial \nu}\, ds,$$

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Is k uniquely determined by Q_k ?.

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- $M : k \to Q_k$ is bounded and analytic in $L^{\infty}_{>0}(\Omega)$;
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✓Sylvester and Uhlmann, 1987 ([16]): k sufficiently smooth $\Rightarrow$ Q_k uniquely determines k. More precisely, they showed that the mapping M is injective over $C^\infty(\overline{\Omega}) \cap L^\infty_{>0}(\Omega)$.

Determining k from knowledge of u on the whole Ω

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Problem of recovering k from information about u:

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Although the mathematical theory of elliptic equations with discontinuous principal coefficients is well known ([14], [12]), there is not much done on the inverse problem of recovering k in these cases.

A regularized variational approach

Setting: Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ a bounded open set with smooth boundary $\Gamma = \partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, with $\Gamma_D \cap \Gamma_N = \emptyset$, $c, k, f, g, h \in L^2(\Omega)$ with $0 < \gamma_1 \leq k(x) \leq \gamma_2$ and $c \geq 0$.

Problem:

$$\mathcal{P} = \mathcal{P}(k, c, f, g, h) : \begin{cases} -\mathrm{div}(k(x)\nabla u(x)) + c(x)u(x) = f(x), & x \in \Omega, \qquad (a) \\ u(x) = g(x), & x \in \Gamma_D, \qquad (b) \\ k(x)\,\nabla u(x) \cdot \vec{n} = h(x), & x \in \Gamma_N. \qquad (c) \end{cases}$$

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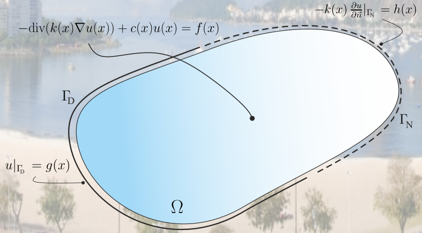

Schematic representation of problem $\mathcal{P}(k, c, f, g, h)$

Assume $g \in C(\Gamma_D)$ and define $H^1_{\Gamma_D,g}(\Omega) \doteq \left\{ v \in H^1(\Omega) \ : \ v|_{\Gamma_D} = g \right\}$.
Multiplying (a) by $v \in H^1_{\Gamma_D,0}(\Omega)$ and integrating we obtain:

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$$0 = \int_\Omega \left(\langle k\nabla u, \nabla v\rangle + cuv \right) \, dx - \int_\Omega fv \, dx - \int_{\Gamma_N} hv \, ds$$

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$$\begin{aligned} 0 &= \int_\Omega \left(\langle k\nabla u, \nabla v\rangle + cuv \right)\, dx - \int_\Omega fv\, dx - \int_{\Gamma_N} hv\, ds \\ &\doteq F(u,v). \end{aligned}$$

Variation formulation of $\mathcal{P}$:

$VF(\mathcal{P})$: Find u in $H^1_{\Gamma_D,g}(\Omega)$ such that $F(u,v) = 0$ for all $v \in H^1_{\Gamma_D,0}(\Omega)$, i.e.

$$\int_\Omega \left(\langle k\nabla u, \nabla v\rangle + cuv \right)\, dx \;=\; \int_\Omega fv\, dx + \int_{\Gamma_N} hv\, ds, \qquad \text{for all } v \in H^1_{\Gamma_D,0}(\Omega).$$

Assume $g \in C(\Gamma_D)$ and define $H^1_{\Gamma_D,g}(\Omega) \doteq \left\{ v \in H^1(\Omega) \;:\; v|_{\Gamma_D} = g \right\}$.
Multiplying (a) by $v \in H^1_{\Gamma_D,0}(\Omega)$ and integrating we obtain:

$$0 = \int_\Omega \left(\langle k\nabla u, \nabla v\rangle + cuv \right) \, dx - \int_\Omega fv \, dx - \int_{\Gamma_N} hv \, ds$$
$$\doteq F(u,v).$$

Variation formulation of $\mathcal{P}$:

$VF(\mathcal{P})$: Find u in $H^1_{\Gamma_D,g}(\Omega)$ such that $F(u,v) = 0$ for all $v \in H^1_{\Gamma_D,0}(\Omega)$, i.e.

$$\int_\Omega \left(\langle k\nabla u, \nabla v\rangle + cuv \right) \, dx \; = \int_\Omega fv \, dx + \int_{\Gamma_N} hv \, ds, \quad \text{for all } v \in H^1_{\Gamma_D,0}(\Omega).$$

Define $B_{k,c} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$B_{k,c}(u,v) \;\doteq\; \int_\Omega \left(\langle k\nabla u, \nabla v\rangle + cuv \right) \, dx.$$

Then $B_{k,c}$ defines an inner product on $H^1(\Omega)$ with associated norm

$$\|u\|^2_{B_{k,c}} \;\doteq\; \int_\Omega \left(k\|\nabla u\|^2 + c|u|^2 \right) \, dx.$$

Define also the energy functional $J : H^1_{\Gamma_D,g}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} J(v) &\doteq \frac{1}{2} B_{k,c}(v,v) - \int_\Omega fv \, dx - \int_{\Gamma_N} hv \, ds \\ &= \frac{1}{2} \int_\Omega \left(\langle k \nabla v, \nabla v \rangle + c\, v^2 \right) \, dx - \int_\Omega fv \, dx - \int_{\Gamma_N} hv \, ds. \end{aligned}$$

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Then

Lemma

For any $u \in H^1_{\Gamma_D,g}(\Omega)$ and any $v \in H^1_{\Gamma_D,0}(\Omega)$ there holds

$$\frac{d}{dt} J(u+tv)\Big|_{t=0} = F(u,v),$$

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and finally

Theorem

Problem $VF(\mathcal{P})$ does have a unique solution $u^* \in H^1_{\Gamma_D,g}(\Omega)$, characterized by the unique minimizer of the energy functional J, i.e.

$$u^* = \operatorname{argmin}_{u \in H^1_{\Gamma_D,g}(\Omega)} J(u).$$

The inverse problem

Given Ω, Γ_D, Γ_N, c, f, g and h, and a prescribed temperature distribution $\hat{u} \in H^1_{\Gamma_D,g}(\Omega)$ find the corresponding distributed conductivity field $k(\cdot)$ such that $u^* = \hat{u}$. That is, “invert” problem $\mathcal{P} = \mathcal{P}(k, c, f, g, h)$ with respect to k.

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For simplicity $n = 2$, and $f = h \equiv 0$. Thus given $\hat{u}(x, y) \in H^1_{\Gamma_D,g}(\Omega)$ we want to find $k = k(x, y)$ such that $\hat{u}$ be the unique solution of problem $\mathcal{P}(k, c, f, g, h)$. Then $k(x, y)$ must satisfy:

$$
\begin{aligned}
0 &= \frac{d}{dt} J(\hat{u} + tv)|_{t=0} \\
&= \int_\Omega \left(\langle k \nabla \hat{u}, \nabla v\rangle + c\hat{u}v\right)\, dx\, dy, \qquad \forall v \in H^1_{\Gamma_D,0}(\Omega).
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Discretization

Let Ω_i, $1 \leq i \leq L$, be a partition of Ω by open sets. For each i let $(x_i, y_i) \in \Omega_i$ and for any function $q(x, y)$ defined on Ω let us denote with q_i the value of q at the point (x_i, y_i), i.e. $q_i \doteq q(x_i, y_i)$. The optimality equation reads:

$$0 = \sum_{i=1}^{L} \left[k_i \left(\hat{u}_{x,i} v_{x,i} + \hat{u}_{y,i} v_{y,i}\right) + c_i \hat{u}_i v_i\right] m(\Omega_i), \qquad \forall v \in H^1_{\Gamma_D,0}(\Omega),$$

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or, assuming a regular partition so that $m(\Omega_i)$ is constant

$$\sum_{i=1}^{L} k_i \left(\hat{u}_{x,i} v_{x,i} + \hat{u}_{y,i} v_{y,i}\right) = -\sum_{i=1}^{L} c_i \hat{u}_i v_i, \qquad \forall v \in H^1_{\Gamma_D,0}(\Omega).$$

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Consider now a finite, arbitrary set of functions

$$v^r \in H^1_{\Gamma_D,0}(\Omega), \qquad 1 \leq r \leq R.$$

Then we must have

$$\sum_{i=1}^{L} k_i \left(\hat{u}_{x,i} v^r_{x,i} + \hat{u}_{y,i} v^r_{y,i}\right) = -\sum_{i=1}^{L} c_i \hat{u}_i v^r_i, \qquad \forall 1 \leq r \leq R.$$

Discretization

By defining

$$a_{r\ell} \doteq \hat{u}_{x,\ell}\, v^r_{x,\ell} + \hat{u}_{y,\ell}\, v^r_{y,\ell} \text{ and } f_r \doteq -\sum_{i=1}^{L} c_i\, \hat{u}_i\, v^r_i,\ 1 \leq \ell \leq L,\ \ 1 \leq r \leq R,$$

we end up with

$$\sum_{\ell=1}^{L} a_{r\ell} k_\ell = f_r, \qquad \forall\, 1 \leq r \leq R,$$

or simply

$$\boxed{\mathbf{A}\, K = F,} \qquad (**)$$

where $\mathbf{A} \in \mathbb{R}^{R\times L}$, $K \in \mathbb{R}^L$ and $F \in \mathbb{R}^R$.

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Still need to impose the condition that all components of the vector K be bounded between the values γ_1 and γ_2.

Discretization

Idea: solve $(**)$ in the least squares sense, weakly imposing this restriction through a penalizer

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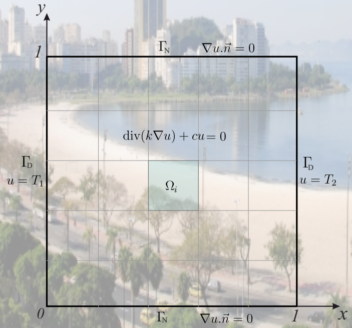
Assumptions and numerical implementation:



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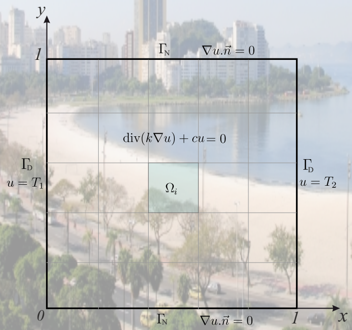

$T_1 > T_2$ and assume $k(x, y)$ can take only two possible values, say k_L and k_U, with $0 < k_L < k_U < \infty$ (only two different materials are present in Ω).

On the penalizers $\mathbf{W}(\mathbf{K})$

At each point $(x, y) \in \Omega$, $k(x, y)$ can only take one of the values k_L or k_U. Then $W(K)$ must be designed so that it deters each and every component of the vector K to take any but one of those two values.

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Option 1: Let $p : \mathbb{R} \to \mathbb{R}$, $p(z) \doteq (z - k_L)(z - k_U) \ = \ z^2 - (k_L + k_U)z + k_L k_U$ and define $W_1 : \mathbb{R}^L \to \mathbb{R}_0^+$ as

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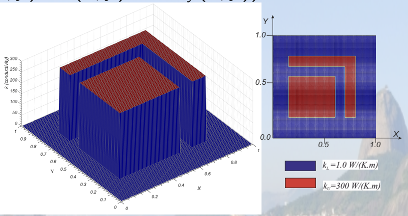
Option 2: Add data-driven information about where to take one or the other value. Let $b \in \mathbb{R}^L$ binary, $b_i = 1$ iif $\|\nabla \hat{u}(x_i, y_i)\| > \gamma$, γ is a given threshold value,

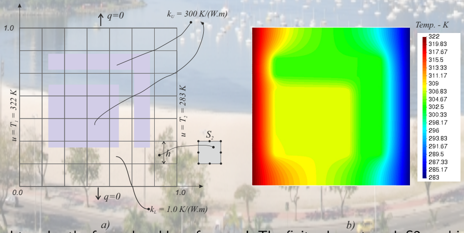
$$W_2(K) \doteq \|b_U \odot (K - k_L\,\mathbf{1})\|^2.$$

Examples and numerical experiments

Case I: We first solved $\mathcal{P}$ with $T_1 = 322\ [K]$, $T_2 = 283\ [K]$, $c(x, y) = 1 =$ const., and $k(x, y)$ as shown (used a standard discretization by FEM, with biquadratic interpolation elements S2 with 8-nodes for computing $\hat{u}(x, y)$, $\hat{u}_x(x, y)$ and $\hat{u}_y(x, y)$)



Conductivity $k(x, y)$ used for solving the forward problem in Case I



a) Sketch of the distretized domain used to solve the forward problem, for case I. The finite element mesh S2 used is regular with elements size $h = 1/200$. b) Temperature distribution $\hat{u}(x, y)$

Case I

Setting 1: Picked $\alpha = 0$ (non-penalized case) and $v^r \in H^1_{\Gamma_D,0}(\Omega)$, $1 \le r \le R$ given by

$$v^{m,n}(x,y) \doteq x^m(1-x)^n, \qquad \text{for } 1 \le m,n \le M,$$

Severe ill-posedness: $Cond(A) \approx 2{,}5 \times 10^{18}$.

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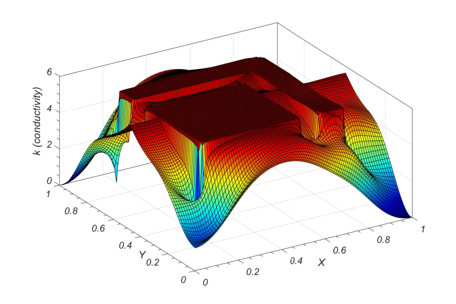

Reconstruction of $k(x,y)$ obtained using a non-penalized least squares approach.

Case I

Setting 2: Take $\alpha > 0$ and $W(K) = W_1(K)$

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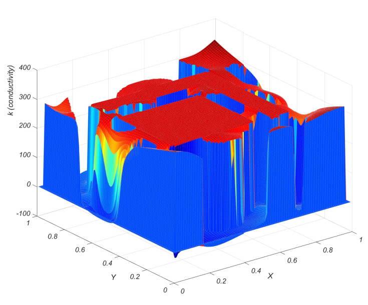

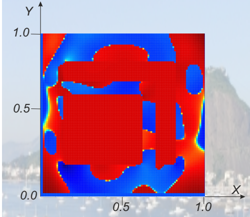

Reconstruction of $k(x, y)$ obtained using $W(K) = W_1(K)$

Case I

Setting 3: Take $\alpha > 0$ and $W(K) = W_2(K)$

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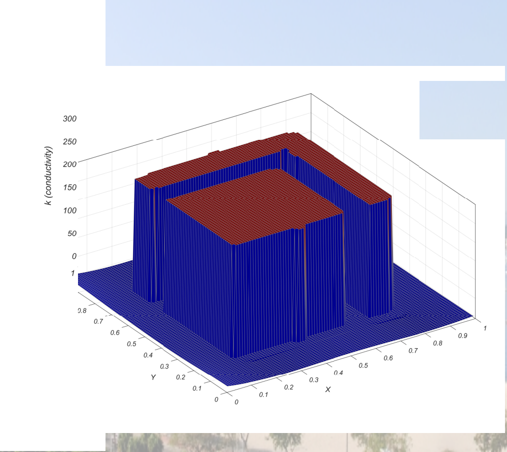
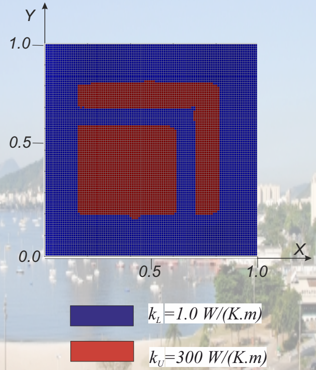
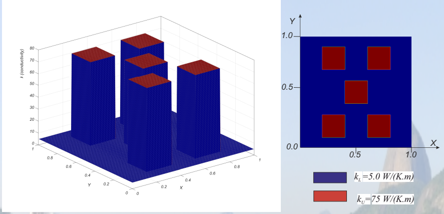




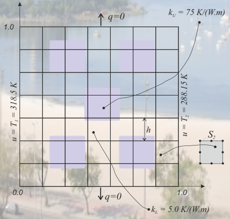
Reconstruction of $k(x, y)$ obtained using $W(K) = W_2(K)$, Case I.

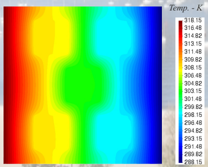
Case II

Conductivity profile shown below; $T_1 = 318{,}15\ [K]$, $T_2 = 288{,}15\ [K]$, $c(x, y) = 1 =$ constant.



Distributed values of the conductivity $k(x, y)$ used for solving the forward problem in Case II





a) Sketch of the distretized domain used to solve the forward problem for Case II. The finite element mesh S2 used is regular with elements size $h = 1/200$. b) Temperature distribution $\hat{u}(x, y)$ for $k(x, y)$ for Case II.

Case II

Setting 3: Take $\alpha > 0$ and $W(K) = W_2(K)$

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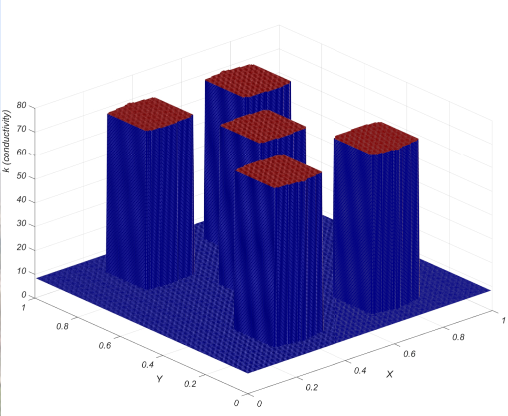
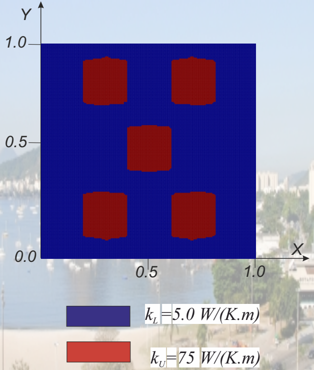
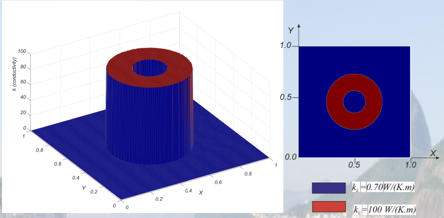




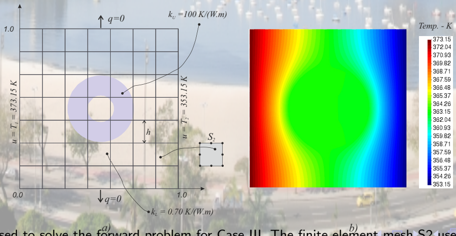
Reconstruction of $k(x, y)$ obtained using $W(K) = W_2(K)$, Case II

Case III

Conductivity profile shown below; $T_1 = 373{,}15\ [K]$, $T_2 = 353{,}15\ [K]$, $c(x,y) = 1{,}0 =$ constant, $k_U = 100$ and $k_L = 0{,}7$.



Distributed values of the conductivity $k(x, y)$ used for solving the forward problem in Case III



a) Sketch of the distretized domain used to solve the forward problem for Case III. The finite element mesh S2 used is regular with elements size

Case III

Setting 3: Take $\alpha > 0$ and $W(K) = W_2(K)$

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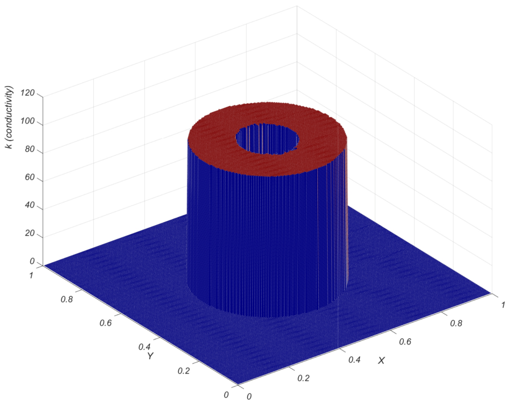
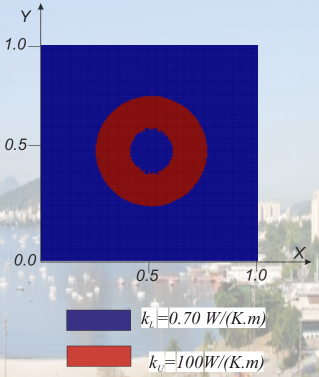
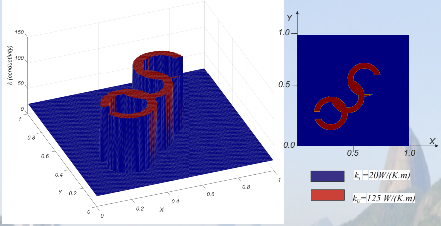




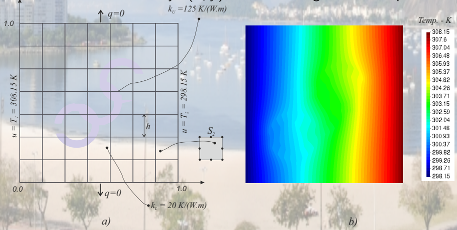
Reconstruction of $k(x, y)$ obtained using $W(K) = W_2(K)$, Case III

Case IV

Conductivity profile shown below; $T_1 = 308{,}15$ $[K]$, $T_2 = 298{,}15$ $[K]$, $c(x,y) = 1{,}0 =$ constant, $k_U = 125$ and $k_L = 20$.



Distributed values of the conductivity $k(x,y)$ used for solving the forward problem in Case IV



a) Sketch of the distretized domain used to solve the forward problem for Case IV. The finite element mesh S2 used is regular with elements size $h = 1/200$. b) Temperature distribution $\hat{u}(x,y)$ for $k(x,y)$ for Case IV.

Case IV

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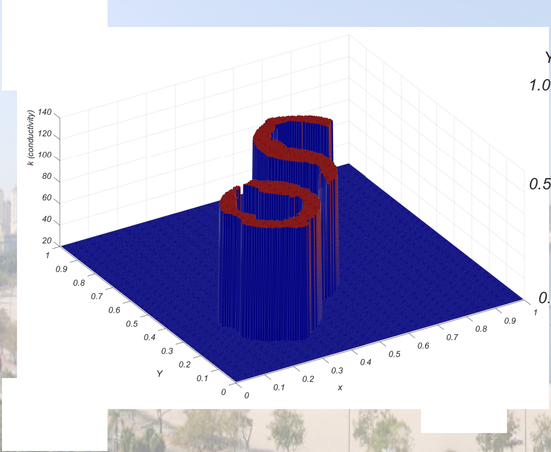

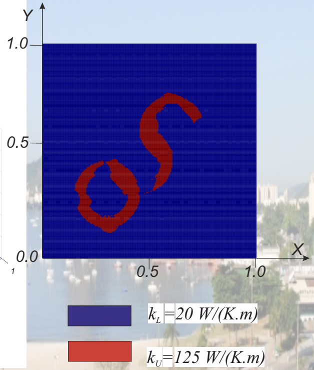

Reconstruction of $k(x, y)$ obtained using $W(K) = W_2(K)$, Case IV

Open problems:

There are many open problems:

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- The case of n materials

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- The case of n materials
- Efficient ways to estimate the threshold parameter γ. For n materials there are $n-1$ threshold parameters.

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thanks for your attention!

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