The application of a time exponential integrator to the wave equations, oriented to seismic imaging

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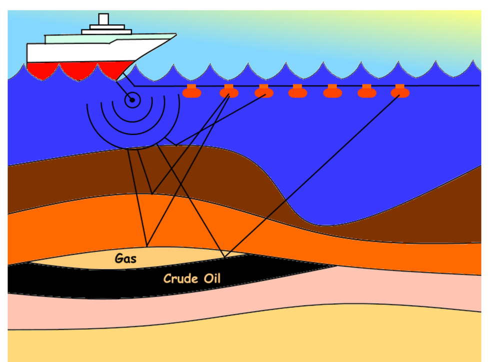
Introduction

The production of images from the subsurface using elastic waves is an important and challenging problem in geophysics [1]. A key stage is the computation of the solution of the wave propagation equations with absorbing boundary conditions (ABC). Encouraging a continously development of innovative numerical methods to approximate its solution, trying to increase the accuracy and efficiency.



A particular class of methods, known as time exponential integrators, have been shown to outperform classical schemes in several physics-related differential equations [2]. However, their application in hyperbolic systems with ABC, like the ones arising in seismic imaging, still lacks theoretical and experimental investigations. For these equations, we study one of the most promising exponential integrators schemes, which is based on Faber polynomials.

Background

Exponential integrators are a class of time integrating methods used to solve ordinary differential equations of first order in time,

$$\frac{d\boldsymbol{u}(t)}{dt} = \boldsymbol{H}\boldsymbol{u}(t) + \boldsymbol{f}(t, \boldsymbol{u}(t)), \quad \boldsymbol{u}(t_0) = \boldsymbol{u}_0.$$

where $\boldsymbol{u}(t), \boldsymbol{u}_0 \in \mathbb{C}^n$, $\boldsymbol{H} \in \mathbb{C}^{n\times n}$, $\boldsymbol{f} : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$; through the approximation of the semi-analytic solution of the constants variation formula,

$$\boldsymbol{u}(t) = e^{(t-t_0)\boldsymbol{H}}\boldsymbol{u}_0 + \int_{t_0}^{t} e^{(t-\tau)\boldsymbol{H}}\boldsymbol{f}(\tau, \boldsymbol{u}(\tau))d\tau. \tag{1}$$

Eq. (1) can be further transformed to the expression

$$\boldsymbol{u}(t) = \begin{bmatrix} \boldsymbol{I}_{n\times n} & \boldsymbol{0} \end{bmatrix} e^{(t-t_0)\tilde{\boldsymbol{H}}} \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{e}_p \end{bmatrix},$$

The problem is transformed in calculating the exponential of a matrix.

where $\boldsymbol{e}_p \in \mathbb{R}^p$ is a canonic vector with one in its last element.

Faber polynomials: given a degree j, and a square matrix $\boldsymbol{H}$, Faber's polynomials are defined as $\boldsymbol{F}_j(\boldsymbol{H})$, with

$$\begin{aligned} &\boldsymbol{F}_0(\boldsymbol{H}) = \boldsymbol{I}_{n\times n}, \quad \boldsymbol{F}_1(\boldsymbol{H}) = \boldsymbol{H}/\gamma - c_0\boldsymbol{I}_{n\times n}, \\ &\boldsymbol{F}_2(\boldsymbol{H}) = \boldsymbol{F}_1(\boldsymbol{H})\boldsymbol{F}_1(\boldsymbol{H}) - 2c_1\boldsymbol{I}_{n\times n}, \\ &\boldsymbol{F}_j(\boldsymbol{H}) = \boldsymbol{F}_1(\boldsymbol{H})\boldsymbol{F}_{j-1}(\boldsymbol{H}) - c_1\boldsymbol{F}_{j-2}(\boldsymbol{H}), \quad j \geq 3, \end{aligned}$$

with

$$\gamma = (a+b)/2, \quad c_0 = d/\gamma, \quad c_1 = c_f^2/(4\gamma^2),$$

where the parameters a, b, c_f, and d, are set according to the spectrum of the operator $\boldsymbol{H}$.

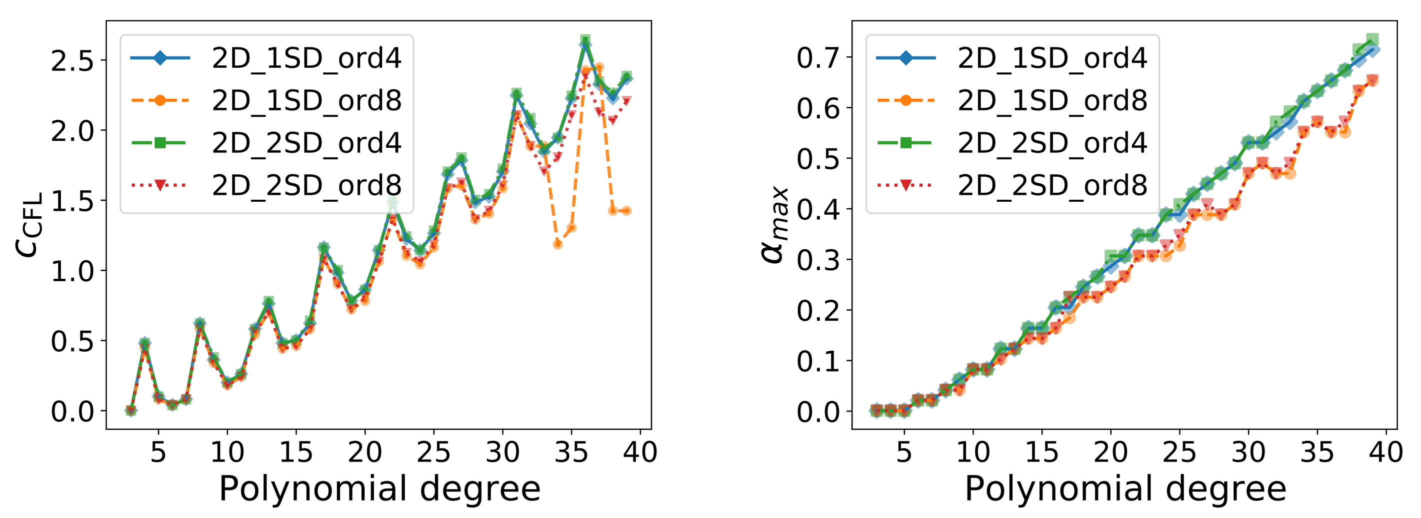
Acoustic **wave equations** with ABC in two dimensions ($(x,y) \in \Omega = [a_1, a_2] \times [b_1, b_2]$, $t > t_0$)

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ w_x \\ w_y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\beta_x\beta_y + c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) & -(\beta_x+\beta_y) & c^2\frac{\partial}{\partial x} & c^2\frac{\partial}{\partial y} \\ (\beta_y - \beta_x)\frac{\partial}{\partial x} & 0 & -\beta_x & 0 \\ (\beta_x - \beta_y)\frac{\partial}{\partial y} & 0 & 0 & -\beta_y \end{pmatrix}\begin{pmatrix} u \\ v \\ w_x \\ w_y \end{pmatrix} + \begin{pmatrix} 0 \\ f \\ 0 \\ 0 \end{pmatrix},$$

,
where $f(t) = (1 - f_0^2\pi^2(t-t_0)^2)e^{-f_0^2\pi^2(t-t_0)^2}$ is a Ricker wavelet in time.

Von Newmann stability and dispersion

Stability and dispersion analysis for the acoustic 2D equations with PML in a homogeneous medium using different spatial discretization orders and equations formulations.

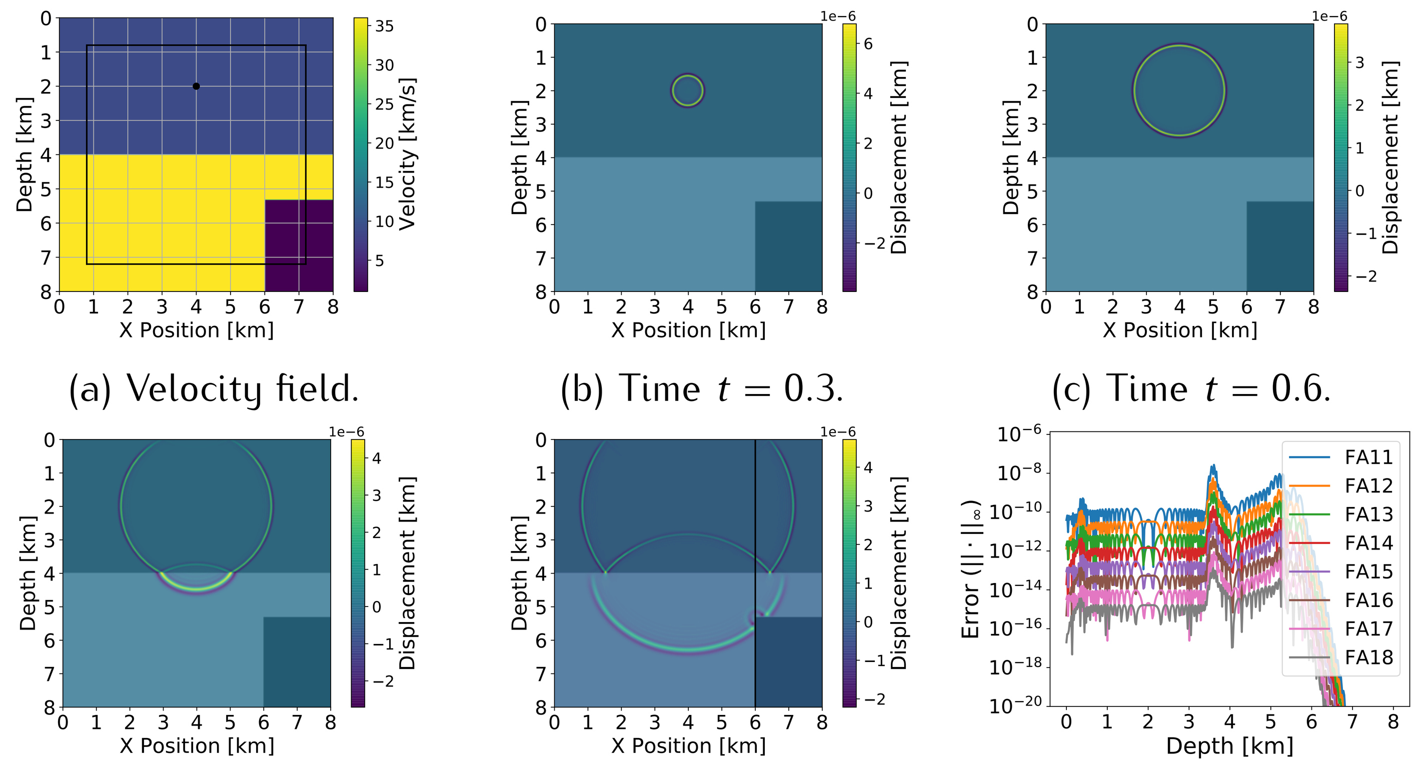


(a) CFL number, $c_{CFL} = c\Delta t_{max}/\Delta x$.

(b) Maximum $\alpha_{max} = c\Delta t_{max}/\Delta x$, with phase change less than 10^{-5}.

High polynomial degrees render larger c_{CFL} and α_{max}.

Corner model

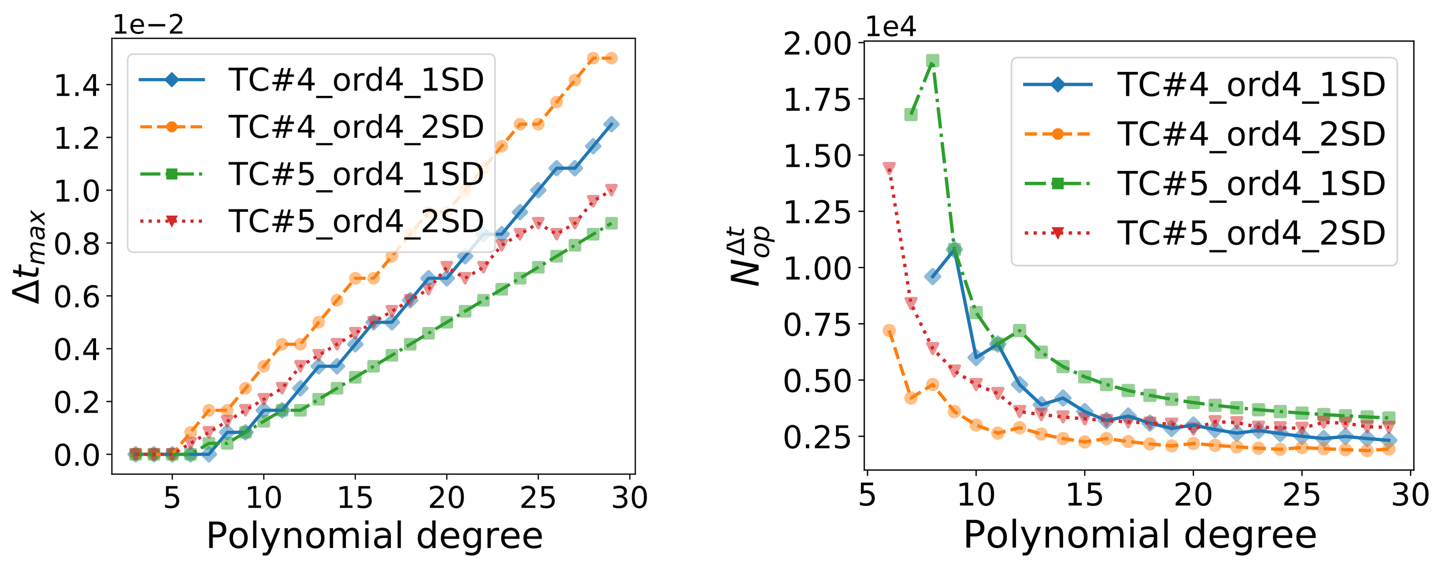


(a) Velocity field. (b) Time $t = 0.3$. (c) Time $t = 0.6$.

(d) Time $t = 0.9$. (e) Time $t = 1.2$. (f) Approximation error at $t = 1.2s$.

When the degree of the Faber polynomial (FA) increases, the error calculated over the black line in Subfigure (e) diminishes by orders of magnitude (Subfigure (f)).

Convergence and efficiency



(a) Maximum time step Δt_{max}, with error less than 10^{-6}

(b) Number of matrix-vector operations ($N_{op}^{\Delta t}$).

Higher polynomial degrees allows larger time steps, and they are more efficient than using low degrees with small time-steps.

Conclusions and future research

- The exponential integrator method based on Faber polynomials can be used to obtain accurate solutions of the wave equations with PML.
- When higher polynomials degrees are used there is an improvement in the stability and dispersion of the scheme.
- Higher degrees permit selecting larger time steps, without loosing accuracy, and this strategy is computationally more efficient.

In the future...

- We will compare this method with other time exponential integrators, high-order Runge-Kuttas with strong stablity preserving properties, and lower order schemes.
- We will use realistic velocity fields to asses the performance of the methods in real-life scenarios.

References

[1] Ikelle LT, Amundsen L (2018). *Introduction to petroleum seismology*. Society of Exploration Geophysicists.
[2] Loffeld J, Tokman M (2013). *Comparative performance of exponential, implicit, and explicit integrators for stiff systems of odes*. Journal of Computational and Applied Mathematics 241:45–67

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