On distinct notions of tensor rank and practical implications to the higher-order completion problem

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Notions of tensor rank

Matrix linear algebra provides powerful tools for organizing and handling data. Mathematical refinements and algorithmic developments resting, for example, upon the Singular Value Decomposition (SVD), have benefited many areas, such as image processing, physics, biomedicine, signal analysis, and neural networks, among others. As data structures become increasingly larger and more complex, more degrees of freedom are required to handle and interpret the data, so the study of multidimensional objects – the so-called tensors – became necessary. In this work, we consider a tensor of order N, i.e., an array of N dimensions [1].

Parallel Factor Decomposition (PARAFAC)

The Parallel Factor Decomposition of a tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ is the combination of R rank-1 components ilustrated in Figure 1:

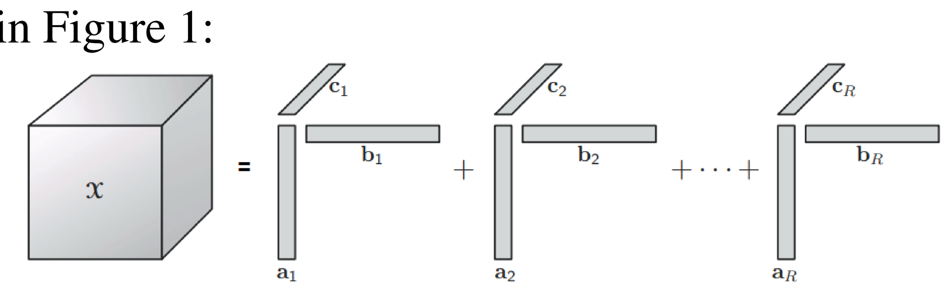

Figure 1: PARAFAC representation of third order tensor [1, Fig 3.1].

The vectors $\boldsymbol{a}_r \in \mathbb{R}^I, \boldsymbol{b}_r \in \mathbb{R}^J$ and $\boldsymbol{c}_r \in \mathbb{R}^K (r = 1, ..., R)$, are the columns of matrices $\boldsymbol{A} \in \mathbb{R}^{I \times R}, \boldsymbol{B} \in \mathbb{R}^{J \times R}$ and $\boldsymbol{C} \in \mathbb{R}^{K \times R}$. We define the tensorial rank as the number R. Given a value that approximates the rank of $\mathcal{X}$, the factors can be determined numerically via Alternated Least Squares (ALS) [1].

High-Order Singular Value Decomposition (HOSVD)

The High Order SVD of a tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ is ilustrated in Figure 2.

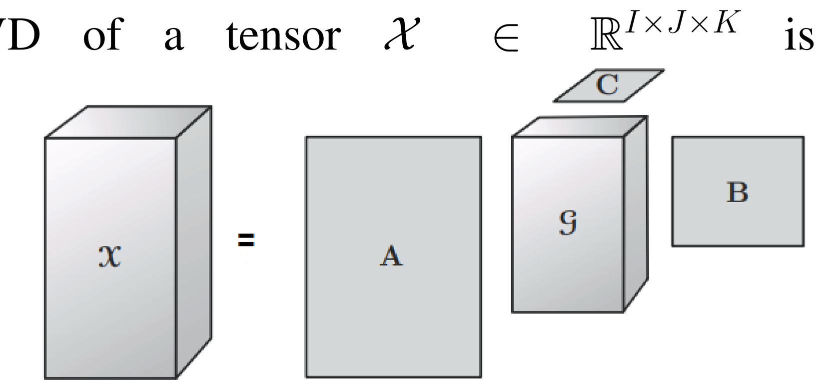

Figure 2: HOSVD representation of third order tensor [1, Fig 4.1].

We have $\boldsymbol{A} \in \mathbb{R}^{I \times P}, \boldsymbol{B} \in \mathbb{R}^{J \times Q}, \boldsymbol{C} \in \mathbb{R}^{K \times R}$ and $\mathcal{G} \in \mathbb{R}^{P \times Q \times R}$ is the tensor core, which stores the interactions between the modes of $\mathcal{X}$. The vector (P, Q, R) is called multidimensional rank and it is related to the ranks of the possible matricizations or unfoldings of $\mathcal{X}$. Given a vector that approximates the multidimensional rank of $\mathcal{X}$, the factors can be determined via SVD factorizations from $\mathcal{X}$ unfoldings or via ALS as well.

Tensor Train Decomposition (TT)

The aforementioned decompositions suffer from serious drawbacks such as ill conditioning during the factors computation process, in the case of PARAFAC, and exponential growth in the number of parameters to represent the tensor, for the HOSVD. The *Tensor Train (TT) decomposition*, together with the *TT-rank* [2], does not suffer from these drawbacks, as its number of estimated parameters grows linearly with the dimensions.

We say $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is in the TT format if its entries are given by

$$x_{i_1, i_2, ..., i_N} = \sum_{\alpha_0=1}^{r_0} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_N=1}^{r_N} g^{(1)}_{\alpha_0, i_1, \alpha_1} g^{(2)}_{\alpha_1, i_2, \alpha_2} \cdots g^{(N)}_{\alpha_{N-1}, i_N, \alpha_N}. \tag{1}$$

The third-order tensors $\mathcal{G}^{(k)} \in \mathbb{R}^{r_{k-1} \times I_k \times r_k}$ are the TT-cores, and the numbers r_k are the TT-ranks, for $k = 0, ..., N$. Note that, by definition, $r_0 = r_N = 1$.

The Completion Problem

The completion problem consists of recovering a set of data from a sample of its entries. It rests upon tools that can capture both global and local information from the data structure, as well as the possible correlations of the entries. In the matrix case, the rank is a powerful tool. However, the notion of rank for high-order structures, e.g. a color image is not unique.

Let $\mathcal{T}$ be the original tensor and $\mathcal{P}_\Omega$ the orthogonal projection operator into the set of indexes Ω which contain the known entries. The classic formulation is:

$$\min_{\mathcal{X}} \{\text{rank}(\mathcal{X}) \mid \mathcal{P}_\Omega(\mathcal{X}) = \mathcal{P}_\Omega(\mathcal{T})\}. \tag{2}$$

Problem (2) can be solved using convex optimization techniques. We addressed the formulation:

$$\min\ f\left(\mathcal{G}^{(1)}, \mathcal{G}^{(2)}, ..., \mathcal{G}^{(N)}\right) = \frac{1}{2} ||\mathcal{P}_\Omega(\mathcal{T}) - \mathcal{P}_\Omega(\mathcal{X})||_F^2. \tag{3}$$

where $||\cdot||_F$ is the Frobenius norm. Note that f is a smooth function on the TT-cores. The solution is obtained by methods based on the steepest descent direction. In this case, the TT-rank plays an important role, because it defines the dimension of the objective function to be minimized and, hence, the computational cost. We have developed a scheme called Tensor Train Weighted Optimization with Dynamical Updating (TT-WOPT-DU) of the TT-rank inspired by ideas from [4, 5].

Algorithm 1: TT-WOPT-DU

```
Data: TT-rank = [1, 1, ..., 1], r_max. Return: G^(1), ..., G^(N)
Initialize TT-cores G^(1), ..., G^(N).
for k = 2, ..., r_max do
    for j = 1, ..., N − 1 do
        if r_j ≥ r_max then
            Move to the next entry of TT-rank.
        else
            TT-rank = [r_1, ..., r_j + 1, ..., r_{N−1}]
            Update the TT-cores by increasing their dimensions but not changing the current representation and solve (3).
            If the quality of the new approximation is not sufficient, reverse the step. Otherwise, accept the increment.
        end
    end
end
```

Results and Conclusions

Let us consider a colored image of dimensions $256 \times 256 \times 3$. In Figure 3, we exhibit the reconstruction of an image with the missing rate of 0.7, and entries are removed uniformly without replacement. The solutions were obtained from an implementation of the methods SiLRTC and TMac-TT (see [3]) in `Julia` language available in a `Github` repository (https://github.com/JoaoLuiz87/completamento_dissertacao_mestrado).

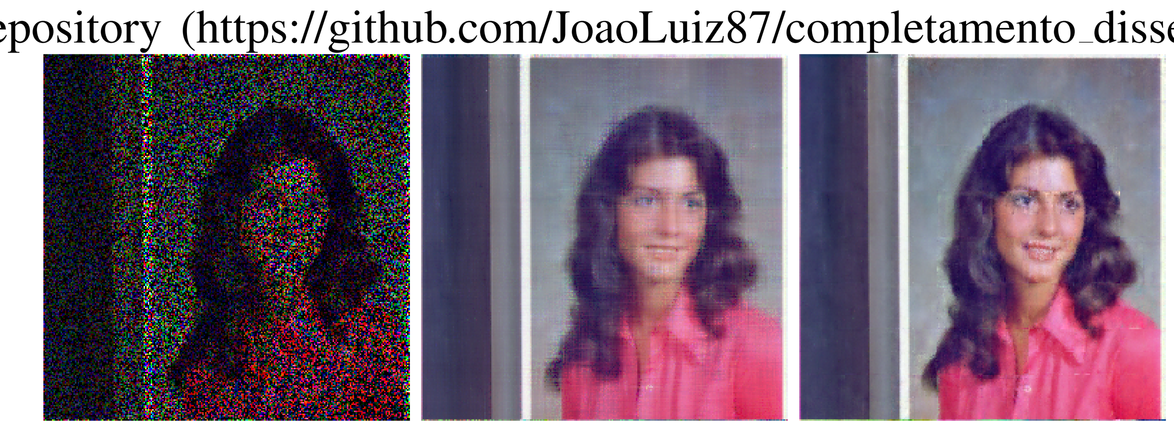

Figure 3: In sequence, we have (i) an image with 70% of missing entries; (ii) solution of (2) obtained by SiLRTC and based on the multilinear rank; (iii) solution of (2) obtained by TMac-TT and based on the TT-rank.

Let $f : [0, 1]^4 \to \mathbb{R}$ given by $f(\boldsymbol{x}) = \exp(-||\boldsymbol{x}||_F)$. We sample the function on a mesh of $20^4 = 160000$ points and consider a 0.95 missing rate. This is a high-order problem and the PARAFAC and HOSVD representations pose difficulties when computing its factors [3]. Hence, we apply the method presented by Algorithm 1. By fixing the first two dimensions and taking slices from the corresponding tensor, it is possible to display the signal as in Figure 4.

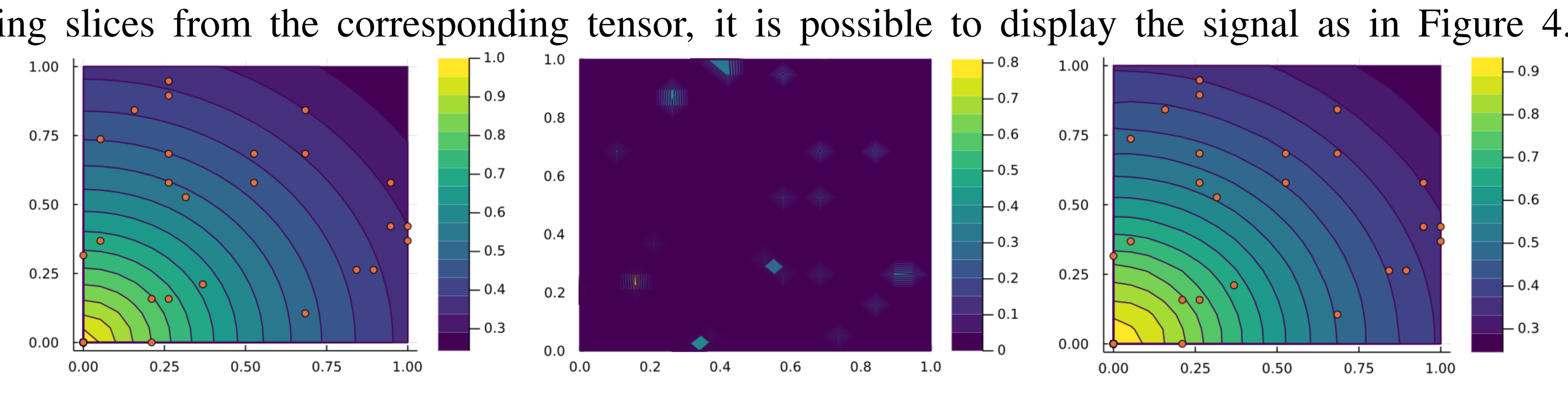

Figure 4: In sequence, we have the contour plots (i) of the original signal; (ii) after 95% of the entries were removed ; (iii) of the recovered signal. The dots in red represent the known entries.

Summing up, the method based on TT-rank formulation was able to successfully recover the multidimensional signal. Thus, tensor completion has proven to be an efficient way to reconstruct data sets with missing inputs.

References

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