A Fokker-Planck-Based Acceleration Technique for Highly Forward-Peaked Scattering Transport Problems

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Abstract

Transport problems with highly forward-peaked scattering can be difficult to solve numerically with standard iterative methods such as source iteration (SI) or non-accelerated generalize minimal residues method (GMRES). Acceleration techniques such as diffusion synthetic acceleration (DSA) and nonlinear diffusion acceleration (NDA) improve the convergence but still quite inefficient. In this work we propose the Fokker-Planck synthetic acceleration (FPSA) technique that greatly outperform the preceding techniques.

Introduction

Motivation: Usually highly charged particles have short mean free paths and highly forward peaked scattering cross sections. Witch means that in such systems, particles travel a very short path before experience a collision and the scattering collisions have a near singular differential scattering cross section in the forward direction. Such systems can be found in electrons transport. Plasma physics, radiation shielding, X-ray machines design, astrophysics etc are also related to the type of system describe above [3].

The Transport Equation: The transport phenomena can be describe by the Boltzmann transport equation and solved for the transport flux ψ. In the present work we are going to assume a mono-energetic, steady state, in slab geometry transport equation

$$\mu\frac{\partial\psi}{\partial x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \int_{-1}^{1}\Sigma_s(\mu,\mu')\psi(x,\mu')d\mu' + Q(x,\mu), \quad (1)$$

with boundary conditions

$$\begin{cases} \psi(0,\mu) &= \psi_L(\mu) \text{ for } \mu > 0, \\ \psi(X,\mu) &= \psi_R(\mu) \text{ for } \mu < 0. \end{cases} \quad (2)$$

Where $Q(x,\mu)$ is the source function, $\psi(x,\mu)$ is the transport flux about the x position with $\mu = \cos(\theta)$ and θ is the angle between the x-axis and the flux direction of travel. The scattering kernel, $\Sigma_s(\mu,\mu')$, is going to be chosen in a way suitable for highly forward peaked scattering settings.

Background: Highly forward peaked scattering problems are difficult to solve numerically for a number of reasons. Solutions for the Boltzmann equation converge slowly when using conventional methods such as source iteration (SI) or generalize minimal residual method (GMRES). Standard acceleration methods such as diffusion synthetic acceleration (DSA) and non-linear synthetic acceleration (NDA) [2] are quite inefficient in accelerating highly forward peaked problems due to not considering higher order Legendre-moments. Higher order Legendre-moments carry necessary information about the system implying that we need a good low-order approximation for the angular flux that can carry the higher order moments. Fokker-Planck approximation can accelerate up to N moments in highly forward peaked settings, being a suitable choice for the problem.

Problem Formulation

Operator form: We can write equation (1) in it operator form as

$$\mathcal{L}\psi = S\psi + Q, \quad (3)$$

where

$$\begin{cases} \mathcal{L} = \mu\frac{\partial}{\partial x} + \Sigma_t, \\ S = \int_{-1}^{1}(\cdot)\Sigma_s(\mu,\mu')d\mu'. \end{cases} \quad (4)$$

Using the Gauss-Legendre quadrature for the integral term in S and taking the sum up to L terms we get an approximation for the S operator in equation (4) as

$$S \approx \sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{s,l}P_l(\mu)\phi_l(x), \quad (5)$$

where $\Sigma_{s,l}$ is the discretized scattering kernel that involves the l order Legendre polynomial and

$$\phi_l(x) = \int_{-1}^{1}P_l(\mu')\psi(x,\mu')d\mu', \quad (6)$$

where $\phi_l(x)$ is the l order transport moment.

Source Iteration: We can attempt to solve the equation (3) using the approximation in equation (5) for the operator S by the source iteration (SI) method. The (SI) method is derived from equation (3) as

$$\mathcal{L}\psi^{(\ell+1)} = S\psi^{(\ell)} + Q. \quad (7)$$

Usually we take $\psi^{(0)} = 0$. The iteration is executed until some criterion is reach. Source Iteration can be shown to be very inefficient in highly forward peaked scattering settings, so we need to introduce a acceleration technique in such situation.

Synthetic Acceleration: Synthetic acceleration introduce a second, correction, step to the (SI) scheme in (7). The (SA) scheme is:

$$\text{Predict: } \mathcal{L}\psi^{(\ell+\frac{1}{2})} = S\psi^{(\ell)} + Q, \quad (8)$$

$$\text{Correct: } \psi^{(\ell+1)} = \psi^{(\ell+\frac{1}{2})} + \mathcal{P}^{-1}S\left(\psi^{(\ell+\frac{1}{2})} - \psi^{(\ell)}\right), \quad (9)$$

where $\mathcal{P} \approx (\mathcal{L} - S)$ is any approximation for the transport operator $(\mathcal{L} - S)$. Different $\mathcal{P}$ operators give rise to different forms of synthetic acceleration.

Diffusion Synthetic Acceleration (DSA): The diffusion approximation for the transport equation is used as the $\mathcal{P}$ operator in equation (9)

$$\mathcal{P} = \left(\frac{-1}{3(\Sigma_t - \Sigma_{s,1})}\frac{d^2}{dx^2} + \Sigma_a\right)\int_{-1}^{1}(\cdot)d\mu. \quad (10)$$

Fokker-Planck Synthetic Acceleration (FPSA): As the mean scattering cosine angle, $\bar{\mu}$, approaches unity, in other words, under the highly forward peaked condition, the S operator in (6) approaches a simpler operator

$$S \approx \frac{\Sigma_{tr}}{2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}, \quad (11)$$

where

$$\mathcal{L}_{FP} = \frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} \quad (12)$$

is the Fokker-Planck operator or the discrete form of the angular Laplacian operator. The $\mathcal{P}$ operator, in this case, is

$$\mathcal{P} = \left(\mu\frac{\partial}{\partial x} + \Sigma_a - \frac{\Sigma_{tr}}{2}\mathcal{L}_{FP}\right). \quad (13)$$

The Fokker-Planck approximation for the transport operator greatly improve the convergence when compare to (SI), (DSA) and non-accelerated (GMRES). This fact is going to be evident when comparing the iterations count and execution time.

Simulations

Fokker-Planck operator: Using discrete ordinates and the $\mathcal{L}_{FP}$ approximation for the S operator in equation (1) gives us

$$\mu_n\frac{\partial\psi_n(x)}{\partial x} + \Sigma_a\Psi_n(x) = \frac{\Sigma_{tr}}{2}\mathcal{L}_{FP}\psi_n(x) + Q_n(x). \quad (14)$$

We use a moment preserving discretization (MPD) that preserves up to N moments of angular flux for discretizing the $\mathcal{L}_{FP}$ operator as

$$\mathcal{L}_{FP}\psi = V^{-1}LV\psi, \quad (15)$$

where $V_{i,j} = P_{i-1}(\mu_j)w_j$ and $L_{i,i} = -i(i-1)$, $i,j = 1,\ldots,N$.

Spatial discretization: We use a linear-discontinuous finite element method to discretize the domain in contiguous cells: (x_k^L, x_k^R), $k = 1,\ldots,K$. In each cell we can express the angular flux as

$$\psi_k(x) = \psi_k^L\left(\frac{x_k^R - x}{\Delta x_k}\right) + \psi_k^R\left(\frac{x - x_k^L}{\Delta x_k}\right), \quad (16)$$

where $\psi_k^L = \psi(x_k^L)$ and $\psi_k^R = \psi(x_k^R)$. We sweep from left to right and the other way around to find ψ_k^L and ψ_k^R depending on the flux direction of travel as can be seeing in Figure 1.

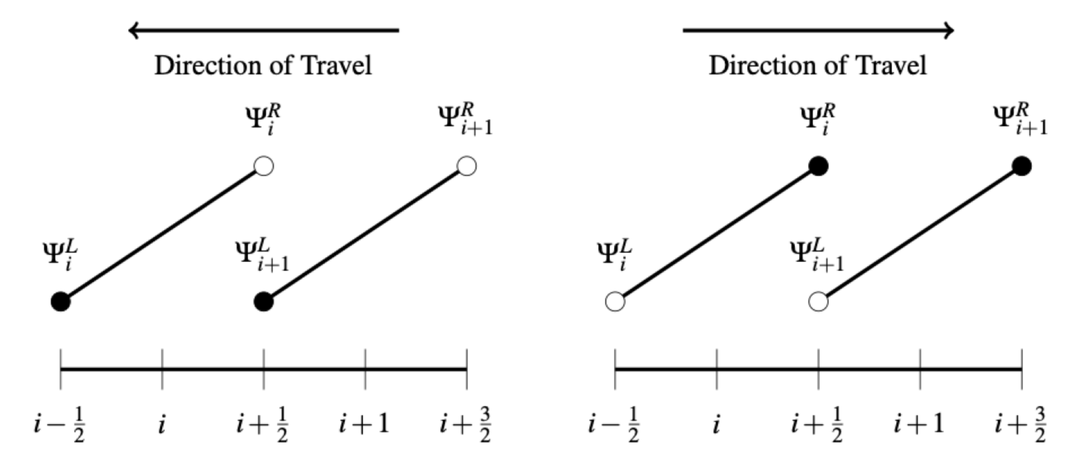

Figure 1: Spatial Discretized Scheme from [1].

Problem Set: Suppose a homogeneous slab with $K = 200$ spatial cells, slab width $X = 400cm$, $\Sigma_a = 0$, $\Sigma_t = \Sigma_{s,0}$, $L = 15$ and $N = 16$. The problem has vacuum boundary conditions and a isotropic source $Q(x) = 0.5$. We use the screened Rutherford Kernel (SRK) best known for modeling scattering behavior of electrons [2].

$$\Sigma_{s,l}^{SRK} = \Sigma_s\int_{-1}^{1}P_l(\mu)\frac{\eta(\eta+1)}{(1+2\eta-\mu)^2}d\mu, \quad (17)$$

as η goes to zero.

Table 1: Numerical results using (SRK) with $\eta = 10^{-5}$ from [1].

Solver	Runtime (s)	Iterations
DSA	2380	53585
GMRES	143	2210
FPSA	1.21	26
FPSA-GMRES	0.589	11

Conclusions and Future Work

We have given a brief introduction to the problem with highly forward peaked scattering settings and a general overview for the classical methods to solve the transport equation. We have successfully establish the (FPSA) method and shown that it greatly outperform (Table 1) the classical methods. In the future we want to simulate problems with heterogeneous slab geometry and different scattering kernels.

References

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